Sharing Stars

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We study here an alternative representation of the Sharing domain [2, 3] which is more efficient at a low cost in loss of precision. In particular, we define a new domain which is equivalent in precision but incorporates a representation which can be more efficient for analysis.

1 Preliminaries

Let V be a set of variables of interest (e.g., the variables of a program). Let $Term$ denote the set of terms over V. Let $\wp^0(S)$ denote the *proper powerset* of set S, i.e., $\wp^0(S) = \wp(S) \setminus \{\emptyset\}$.

A *sharing group* is a set of variables of interest: it represents the possible sharing among its variables. Let $SG = \wp^0(V)$ be the set of all sharing groups. A *sharing set* is a set of sharing groups. The Sharing domain is $SH = \wp(SG)$, the set of all sharing sets.

That a sharing group represents the possible sharing of its variables means the following. Let a concrete substitution μ be approximated by a Sharing abstract substitution $\mu^\alpha \in SH$, if all terms $x\mu$ for every $x \in S \subseteq V$ have at least one variable in common which does not occur in any term $y\mu$ for any variable $y \notin S$ then $S \in \mu^\alpha$. Note that it is *possible* sharing because if $S \in \mu^\alpha$ then there may or may not be common variables exclusive to the terms $x\mu$. However, the counterpart of sharing, i.e., independence, is definite information, because if $S \notin \mu^\alpha$ then there is no common variable exclusive to the terms $x\mu$.

For two sharing sets $s_1 \in SH$, $s_2 \in SH$, let $s_1 \uplus s_2$ be their *binary union*, i.e., the result of applying union to the two elements in each pair in the cartesian product $s_1 \times s_2$. Let also s_1^* be the *star union* of s_1, i.e., its closure under union.

Given terms s and t, and sharing set $sh \in SH$, we denote by sh_t the set of sharing groups in sh which have non-empty intersection with the set of variables of t. By extension, in sh_{st} st acts as a single term. Also, $\overline{sh_t}$ is the complement of sh_t, i.e., $sh \setminus sh_t$.

Abstract unification for equation $x = t$, where x is a variable and t a term, in a store represented by abstract substitution $sh \in SH$, is defined in Sharing as [1]:[1]

$$amgu(x = t, sh) = \overline{sh_{xt}} \cup (sh_x^* \bowtie sh_t^*)$$

2 Star Sets

We will call $sh \in SH$ a *star set* when we use it to represent its own closure under union. Let $SS = SH$ denote the set of all star sets. A star set $ss \in SS$ represents the sharing set $ss^* \in SH$. When a sharing set $sh \in SH$ includes the closure under union of some other sharing set ss, the representation can be simplified by using ss as a star set. Thus, if $sh = sh' \cup ss^*$, then we can use the pair (ss, sh') to represent sh. Note that the overall sharing represented does not change. Thus, we define $SSH = \{(ss, sh) \mid ss \in SS, sh \in SH\}$ and interpret an element $(ss, sh) \in SSH$ as the corresponding element $(ss^* \cup sh) \in SH$.

The operations of binary and star union carry over straightforwardly to star sets. The question is: do they convey any loss of precision? The answer is: no. I.e., the result of a binary or star union operation over star sets conveys the same sharing than the corresponding operation over the closures of the star sets. This is not, however, the case for set union.

Lemma 1 *Let* $ss_1 \in SS$, $ss_2 \in SS$, *then*

$$ss_1^* = (ss_1^*)^* \tag{1}$$

$$(ss_1 \bowtie ss_2)^* = ss_1^* \bowtie ss_2^* \tag{2}$$

Proof (1) holds trivially, since * is a closure.

To show (2) we first show that $(ss_1 \bowtie ss_2)^* \subseteq ss_1^* \bowtie ss_2^*$. Let $s \in (ss_1 \bowtie ss_2)^*$, then there is $\{s_1, \ldots, s_n\} \subseteq (ss_1 \bowtie ss_2)$, $n \geq 1$, such that $s = \cup_{i=1}^n s_i$. So there are also $\{a_1, \ldots, a_n\} \subseteq ss_1$ and $\{b_1, \ldots, b_n\} \subseteq ss_2$ such that $s_i = a_i \cup b_i$ for each $i = 1, \ldots, n$. Therefore, $(\cup_{i=1}^n a_i) \in ss_1^*$ and $(\cup_{i=1}^n b_i) \in ss_2^*$; from which we have $(\cup_{i=1}^n a_i) \cup (\cup_{i=1}^n b_i) \in (ss_1^* \bowtie ss_2^*)$. Since $(\cup_{i=1}^n a_i) \cup (\cup_{i=1}^n b_i) = \cup_{i=1}^n (a_i \cup b_i) = \cup_{i=1}^n s_i = s$, then $s \in (ss_1^* \bowtie ss_2^*)$.

[1] Note that $sh_t^* = (sh_t)^*$.

We now show that $ss_1^* \bowtie ss_2^* \subseteq (ss_1 \bowtie ss_2)^*$. Let $s \in (ss_1^* \bowtie ss_2^*)$, then there are $s_1 \in ss_1^*$ and $s_2 \in ss_2^*$, such that $s = s_1 \cup s_2$. So there are also $\{a_1, \ldots, a_m\} \subseteq ss_1$ and $\{b_1, \ldots, b_n\} \subseteq ss_2$, $m \geq 1$, $n \geq 1$, such that $s_1 = \cup_{i=1}^m a_i$ and $s_2 = \cup_{i=1}^n b_i$. Thus, for any $i = 1, \ldots, m$ and $j = 1, \ldots, n$, $a_i \cup b_j \in (ss_1 \bowtie ss_2)$; which means that $\cup_{i=1}^m{}_{j=1}^n (a_i \cup b_j) \in (ss_1 \bowtie ss_2)^*$. Since $\cup_{i=1}^m{}_{j=1}^n (a_i \cup b_j) = (\cup_{i=1}^m a_i) \cup (\cup_{i=1}^n b_i) = s_1 \cup s_2 = s$, then $s \in (ss_1 \bowtie ss_2)^*$.

Lemma 2 *Let $ss_1 \in SS$, $ss_2 \in SS$, then*

$$(ss_1 \cup ss_2)^* \supseteq ss_1^* \cup ss_2^* \tag{3}$$

but not the other way around.

Proof Let $s \in (ss_1^* \cup ss_2^*)$, then either $s \in ss_1^*$ or $s \in ss_2^*$ or both. Let, without loss of generality, $s \in ss_1^*$. Then there is $\{s_1, \ldots, s_n\} \subseteq ss_1$, $n \geq 1$, such that $s = \cup_{i=1}^n s_i$. Thus, $\{s_1, \ldots, s_n\} \subseteq (ss_1 \cup ss_2)$, and therefore $\cup_{i=1}^n s_i \in (ss_1 \cup ss_2)^*$. Since $\cup_{i=1}^n s_i = s$, then $s \in (ss_1 \cup ss_2)^*$.

To see that the other direction does not hold in general, take[2] $ss_1 = \{vw\}$ and $ss_2 = \{xy\}$. We have $ss_1^* = ss_1 = \{vw\}$ and $ss_2^* = ss_2 = \{xy\}$, so that $ss_1^* \cup ss_2^* = \{vw, xy\}$. However, $(ss_1 \cup ss_2)^* = \{vw, vwxy, xy\}$.

Note that property (2) above can be used to conveniently redefine *amgu* to be used with star sets. To be able to do this we need to define the counterpart of sh_t, the restriction of a sharing set sh on a term t, in an adequate way: we should be able to guarantee at least correctness of the redefined amgu, if not precision. Note that to plainly carry over the restriction operation to star sets is not correct:

Lemma 3 *Let $ss \in SS$ and $t \in Term$,*

$$ss_t^* \subseteq (ss^*)_t \tag{4}$$

but not the other way around.

Proof Let $s \in ss_t^*$, then there is $\{s_1, \ldots, s_n\} \subseteq ss_t$, $n \geq 1$, such that $s = \cup_{i=1}^n s_i$. Thus, $\{s_1, \ldots, s_n\} \subseteq ss$ and[3] $s_i \cap t \neq \emptyset$ for

[2] To simplify notation, we denote a sharing group by the concatenation of its variables, e.g., xyz is $\{x, y, z\}$.

[3] To simplify notation, we abuse term t to denote its own set of variables.

all $i = 1, \ldots, n$. So $(\cup_{i=1}^n s_i) \in ss^*$ and $(\cup_{i=1}^n s_i) \cap t \neq \emptyset$, which means that $(\cup_{i=1}^n s_i) \in (ss^*)_t$. Since $\cup_{i=1}^n s_i = s$, then $s \in (ss^*)_t$.

To see that the other direction does not hold in general, take $ss = \{x, y\}$ and $t = x$. We have $ss_t = \{x\} = ss_t^*$. However, $ss^* = \{x, xy, y\}$ and $(ss^*)_t = \{x, xy\}$.

To solve the problem pointed above we define $| : SS \times Term \to SS$ as follows. For any $ss \in SS$ and term t:

$$ss|_t = ss_t \cup (ss_t \bowtie \overline{ss_t})$$

The operation lifts naturally to $ss|_{rt}$ for two terms r and t. We have that the restriction $ss|_t$ of a star set ss does not lose precision w.r.t. the original restriction operation of the sharing set corresponding to ss.

Lemma 4 *Let $ss \in SS$ and $t \in Term$,*

$$(ss|_t)^* = (ss^*)_t \tag{5}$$

$$\overline{ss_t}^* = \overline{(ss^*)_t} \tag{6}$$

Proof First note that
$(ss|_t)^* = (ss_t \cup (ss_t \bowtie \overline{ss_t}))^* \supseteq ss_t^* \cup (ss_t \bowtie \overline{ss_t})^* = ss_t^* \cup (ss_t^* \bowtie \overline{ss_t}^*)$.

Now we prove that $(ss|_t)^* \supseteq (ss^*)_t$. Let $s \in (ss^*)_t$, so that $s \in ss^*$ and $s \cap t \neq \emptyset$. Then there are $\{s_1, \ldots, s_m\} \subseteq ss$, $m \geq 0$, and $\{t_1, \ldots, t_n\} \subseteq ss$, $n \geq 1$, such that $s = (\cup_{i=1}^m s_i) \cup (\cup_{i=1}^n t_i)$, $s_i \cap t = \emptyset$ for all $i = 1, \ldots, m$, and $t_j \cap t \neq \emptyset$ for all $j = 1, \ldots, n$.

Consider $m = 0$. Then $s = \cup_{i=1}^n t_i$ and $\{t_1, \ldots, t_n\} \subseteq ss_t$. Thus, $\cup_{i=1}^n t_i = s \in ss_t^*$, and therefore $s \in (ss|_t)^*$.

If $m \geq 1$ we have $\{s_1, \ldots, s_m\} \subseteq \overline{ss_t}$ and $\{t_1, \ldots, t_n\} \subseteq ss_t$. Therefore, $(\cup_{i=1}^m s_i) \in \overline{ss_t}^*$ and $(\cup_{i=1}^n t_i) \in ss_t^*$, so that, $(\cup_{i=1}^m s_i) \cup (\cup_{i=1}^n t_i) = s \in (ss_t^* \bowtie \overline{ss_t}^*)$. Thus, $s \in (ss|_t)^*$.

We now prove that $(ss|_t)^* \subseteq (ss^*)_t$. Let $s \in (ss|_t)^*$. Then there are $\{t_1, \ldots, t_m\} \subseteq ss_t$, $m \geq 0$, and $\{r_1, \ldots, r_n\} \subseteq (ss_t \bowtie \overline{ss_t})$, $n \geq 0$, such that $s = (\cup_{i=1}^m t_i) \cup (\cup_{i=1}^n r_i)$, and not $m = n = 0$. Then we have either $\{t_1, \ldots, t_m\} \subseteq ss$, $m \geq 1$, and $t_i \cap t \neq \emptyset$ for $i = 1, \ldots, m$, or $\{t_{m+1}, \ldots, t_{m+n}\} \subseteq ss_t$, $\{s_1, \ldots, s_n\} \subseteq \overline{ss_t}$, $n \geq 1$, such that $r_j = s_j \cup t_{m+j}$ for $j = 1, \ldots, n$, or both. In the latter case, for all $j = 1, \ldots, n$, $t_{m+j} \cap t \neq \emptyset$ and $s_j \cap t = \emptyset$. In any case, $((\cup_{i=1}^m t_i) \cup (\cup_{i=1}^n (t_{m+i} \cup s_i))) \cap t \neq \emptyset$, so that $s \cap t \neq \emptyset$. Also, $\{s_1, \ldots, s_n, t_1, \ldots, t_{m+n}\} \subseteq ss$, so that $s \in ss^*$. Thus, $s \in (ss^*)_t$.

To prove (6) we first prove $\overline{ss_t}^* \subseteq \overline{(ss^*)_t}$. Let $s \in \overline{ss_t}^*$. Then there is $\{s_1, \ldots, s_n\} \subseteq \overline{ss_t}$, $n \geq 1$, such that $s = \cup_{i=1}^n s_i$. Therefore, $\{s_1, \ldots, s_n\} \subseteq ss$ and $s_i \cap t = \emptyset$ for all $i = 1, \ldots, n$, so that $\cup_{i=1}^n s_i = s \in ss^*$ and $s \cap t = \emptyset$. Thus, $s \in \overline{(ss^*)_t}$.

Finally, we prove $\overline{(ss^*)_t} \subseteq \overline{ss_t}^*$. Let $s \in \overline{(ss^*)_t}$. Then $s \in ss^*$ and $s \cap t = \emptyset$. Therefore, there is $\{s_1, \ldots, s_n\} \subseteq ss$, $n \geq 1$, such that $s = \cup_{i=1}^n s_i$, and $s_i \cap t = \emptyset$ for all $i = 1, \ldots, n$. Thus, $\{s_1, \ldots, s_n\} \subseteq \overline{ss_t}$, so that $\cup_{i=1}^n s_i = s \in \overline{ss_t}^*$.

We can now define abstract unification for SS as follows. For equation $x = t$, where x is a variable and t a term, and $ss \in SS$:

$$amgu^s(x = t, ss) = \overline{ss_{xt}} \cup (ss|_x \bowtie ss|_t)$$

Theorem 1 *Let $ss \in SS$ and equation $x = t$, where $x \in V$ and $t \in Term$,*

$$amgu^s(x = t, ss)^* = amgu(x = t, ss^*) \tag{7}$$

Proof We first show that

$$(ss^*)_t = (ss^*)_t^* \tag{8}$$

Since * is a closure, $(ss^*)_t \subseteq ((ss^*)_t)^* = (ss^*)_t^*$. From (4), $(ss^*)_t^* \subseteq ((ss^*)^*)_t = (ss^*)_t$.

Now we show that $amgu^s(x = t, ss)^* \supseteq amgu(x = t, ss^*)$:

$$\begin{array}{lcll}
amgu^s(x = t, ss)^* & = & (\overline{ss_{xt}} \cup (ss|_x \bowtie ss|_t))^* & \\
& \supseteq & \overline{ss_{xt}}^* \cup (ss|_x \bowtie ss|_t)^* & \text{by (3)} \\
& = & \overline{ss_{xt}}^* \cup ((ss|_x)^* \bowtie (ss|_t)^*) & \text{by (2)} \\
& = & \overline{(ss^*)_{xt}} \cup ((ss^*)_x \bowtie (ss^*)_t) & \text{by Lemma 4} \\
& = & \overline{(ss^*)_{xt}} \cup ((ss^*)_x^* \bowtie (ss^*)_t^*) & \text{by (8)} \\
& = & amgu(x = t, ss^*) &
\end{array}$$

Finally, we show that $amgu^s(x = t, ss)^* \subseteq amgu(x = t, ss^*)$: Let $s \in amgu^s(x = t, ss)^* = (\overline{ss_{xt}} \cup (ss|_x \bowtie ss|_t))^*$. Then there are $\{a_1, \ldots, a_m\} \subseteq \overline{ss_{xt}}$, $m \geq 0$, and $\{b_1, \ldots, b_n\} \subseteq (ss|_x \bowtie ss|_t)$, $n \geq 0$, such that $s = (\cup_{i=1}^m a_i) \cup (\cup_{i=1}^n b_i)$, but not $m = n = 0$.

If $n = 0$ then $s = (\cup_{i=1}^m a_i)$, $m \geq 1$, so that $s \in \overline{ss_{xt}}^* = \overline{(ss^*)_{xt}}$. Let then $n \geq 1$, whether $m = 0$ or $m \geq 1$.

There are $\{c_1,\ldots,c_n\} \subseteq ss|_x$ and $\{d_1,\ldots,d_n\} \subseteq ss|_t$ such that $b_i = c_i \cup d_i$, for all $i = 1,\ldots,n$. Then, for all $i = 1,\ldots,n$, either $c_i \in ss_x$ or $c_i = e_i \cup f_i$, $e_i \in ss_x$, and $f_i \in \overline{ss_x}$, or both. Let, without loss of generality, $c_i = e_i \cup f_i$, with $e_i \in ss_x$, and either $f_i = \emptyset$ or $f_i \in \overline{ss_x}$. The same reasoning can be applied to the $d_i \in ss|_t$, $i = 1,\ldots,n$; so, let $d_i = g_i \cup h_i$, with $g_i \in ss_t$, and either $h_i = \emptyset$ or $h_i \in \overline{ss_t}$.

So, we have $s = (\cup_{i=1}^m a_i) \cup (\cup_{i=1}^n ((e_i \cup f_i) \cup (g_i \cup h_i)))$, $\{e_1,\ldots,e_n\} \subseteq ss_x$, $\{g_1,\ldots,g_n\} \subseteq ss_t$, and, if they exist, $\{a_1,\ldots,a_m\} \subseteq \overline{ss_{xt}}$, $\{f_1,\ldots,f_n\} \subseteq \overline{ss_x}$, $\{h_1,\ldots,h_n\} \subseteq \overline{ss_t}$. Therefore, $\{a_1,\ldots,a_m, e_1,\ldots,e_n,\ f_1,\ldots,f_n,\ g_1,\ldots,g_n,\ h_1,\ldots,h_n\} \subseteq ss$ and for all $i = 1,\ldots,n$, $e_i \cap x \neq \emptyset$ and $g_i \cap t \neq \emptyset$. Then $s \in ss^*$, $s \cap x \neq \emptyset$, and $s \cap t \neq \emptyset$; so that $s \in (ss^*)_x = (ss^*)^*_x$ and $s \in (ss^*)_t = (ss^*)^*_t$. Thus, $s \in ((ss^*)^*_x \bowtie (ss^*)^*_t)$.

Hence, whether $n = 0$ or $n \geq 1$, we have that $s \in (\overline{(ss^*)_{xt}} \cup ((ss^*)^*_x \bowtie (ss^*)^*_t)) = amgu(x = t, ss^*)$.

Note that the above theorem is saying that by using star sets for analysis, instead of (closed) sharing sets, the analysis will not lose precision w.r.t. the Sharing domain (if all sharing sets during analysis were closed under union, obviously). Moreover, in the case of star sets there is no need for the costly star union operation, therefore gaining in analysis efficiency.

3 The Star-Sharing Domain

Despite the above result, for a practical analysis, we will need to use both star sets and sharing sets, i.e., we have to use the domain of pairs SSH. Here, there is the problem of the simultaneous handling of star sets and sharing sets. One issue is how to take into account the crossed effects between the sharing set and the star set parts of a pair. Another issue is how to combine new star sets arising from unification together with the star set of a given pair. There is also the issue of the possible advantages in the representation that transferring information from the sharing set part to the star set part of a pair might give.

We could use $amgu$ for the sharing set part and $amgu^s$ for the star set part. However, we have to address the previous issues. We first define an abstract unification in SSH where we simply use set union for combining star sets. For equation $x = t$, $x \in V$, $t \in Term$, and $(ss, sh) \in SSH$:

$$amgu^\delta(x = t, (ss, sh)) = (\overline{ss_{xt}} \cup ((ss|_x \cup sh_x) \bowtie (ss|_t \cup sh_t)), \overline{sh_{xt}})$$

Note that we address the previously mentioned concerns by accumulating results as much as possible in the star set part of the pair, where star union is not required, seeking thus for efficiency. This can be better seen from the following equivalence, which holds because of distributivity of binary set union w.r.t. set union (ref. (12) below):

$$\begin{array}{c}(ss|_x \cup sh_x) \bowtie (ss|_t \cup sh_t) \\ = \\ (ss|_x \bowtie ss|_t) \cup (ss|_x \bowtie sh_t) \cup (sh_x \bowtie ss|_t) \cup (sh_x \bowtie sh_t)\end{array} \tag{9}$$

Lemma 5 *Let $sh_1 \in SH$, $sh_2 \in SH$, $sh_3 \in SH$, and $t \in Term$, then:*

$$(sh_1 \cup sh_2)_t = sh_{1t} \cup sh_{2t} \tag{10}$$

$$\overline{(sh_1 \cup sh_2)_t} = \overline{sh_{1t}} \cup \overline{sh_{2t}} \tag{11}$$

$$sh_1 \bowtie (sh_2 \cup sh_3) = (sh_1 \bowtie sh_2) \cup (sh_1 \bowtie sh_3) \tag{12}$$

$$(sh_1^* \cup sh_2)^* = (sh_1 \cup sh_2)^* \tag{13}$$

Proof All four results are straightforward to show.

In (9) one can see that $amgu^\delta$ incorporates a term that accounts for the unification within the star set part of the initial pair, another term that accounts for the sharing set part (which is "transferred" to the star set part), and two other terms for the crossed effects between both parts. Abstract unification $amgu^\delta$ is correct (but imprecise!):

Theorem 2 *Let $(ss, sh) \in SSH$ and equation $x = t$, where $x \in V$ and $t \in Term$, and $amgu^\delta(x = t, (ss, sh)) = (ss_\delta, sh_\delta)$,*

$$ss_\delta^* \cup sh_\delta \supseteq amgu(x = t, ss^* \cup sh) \tag{14}$$

but not the other way around.

Proof First note that

$$\overline{ss_{xt}}^* \cup \overline{sh_{xt}} = \overline{(ss^* \cup sh)_{xt}} \tag{15}$$

since $\overline{ss_{xt}}^* \cup \overline{sh_{xt}} = \overline{(ss^*)_{xt}} \cup \overline{sh_{xt}} = \overline{(ss^* \cup sh)_{xt}}$ by (6) and (11), respectively.

Also, note that:

$$(ss|_t \cup sh_t)^* = (ss^* \cup sh)_t^* \tag{16}$$

since $(ss|_t \cup sh_t)^* = ((ss|_t)^* \cup sh_t)^* = ((ss^*)_t \cup sh_t)^* = (ss^* \cup sh)^*_t$ by (13), (5), and (10), respectively.

Now we can write:

$$
\begin{array}{lll}
ss^*_\delta \cup sh_\delta & & \\
= & (\overline{ss_{xt}} \cup ((ss|_x \cup sh_x) \bowtie (ss|_t \cup sh_t)))^* \cup \overline{sh_{xt}} & \\
\supseteq & \overline{ss_{xt}}^* \cup ((ss|_x \cup sh_x) \bowtie (ss|_t \cup sh_t))^* \cup \overline{sh_{xt}} & \text{by (3)} \\
= & \overline{ss_{xt}}^* \cup ((ss|_x \cup sh_x)^* \bowtie (ss|_t \cup sh_t)^*) \cup \overline{sh_{xt}} & \text{by (2)} \\
= & \overline{ss_{xt}}^* \cup ((ss^* \cup sh)^*_x \bowtie (ss^* \cup sh)^*_t) \cup \overline{sh_{xt}} & \text{by (16)} \\
= & \overline{ss_{xt}}^* \cup \overline{sh_{xt}} \cup ((ss^* \cup sh)^*_x \bowtie (ss^* \cup sh)^*_t) & \\
= & \overline{(ss^* \cup sh)_{xt}} \cup ((ss^* \cup sh)^*_x \bowtie (ss^* \cup sh)^*_t) & \text{by (15)} \\
= & amgu(x = t, ss^* \cup sh) &
\end{array}
$$

To see that it is not always the case that $ss^*_\delta \cup sh_\delta \subseteq amgu(x = t, ss^* \cup sh)$ take $ss = \{w\}$ and $sh = \{x, y\}$ with $t = y$. We have that $ss|_x = ss|_t = \overline{sh_{xt}} = \emptyset$, so that:

$$ss^*_\delta \cup sh_\delta = (\overline{ss_{xt}} \cup (sh_x \bowtie sh_t))^*$$

$$amgu(x = t, ss^* \cup sh) = \overline{ss_{xt}}^* \cup ((ss^* \cup sh)^*_x \bowtie (ss^* \cup sh)^*_t)$$

We also have that $ss^* = ss = \overline{ss_{xt}} = \overline{ss_{xt}}^* = \{w\}$, $ssh = ss^* \cup sh = \{w, x, y\}$, $ssh_x = sh_x = \{x\} = ssh^*_x$, $ssh_t = sh_t = \{y\} = ssh^*_t$, and $ssh^*_x \bowtie ssh^*_t = sh_x \bowtie sh_t = \{xy\}$. Thus, we have $ss^*_\delta \cup sh_\delta = \{w, xy\}^* = \{w, wxy, xy\}$ but $amgu(x = t, ss^* \cup sh) = \{w\} \cup \{xy\} = \{w, xy\}$.

Note that the above theorem implies a loss of precision which is due to the fact that set union and closure under union do not commute, as Lemma 2 shows. We can remedy this by avoiding the use of set union, accumulating the different star sets that appear during unification into a set of them, instead of merging them with set union. Thus, we define for each $sss \in \wp(SS)$, $sss' \in \wp(SS)$, $t \in Term$:

$$sss|_t = \{ss|_t \mid ss \in sss\} \qquad \text{and} \qquad \overline{sss|_t} = \{\overline{ss_t} \mid ss \in sss\}$$

The definitions lift naturally to $sss|_{rt}$ and $\overline{sss|_{rt}}$ for two terms r and t. For a set of star sets, the sharing that it represents corresponds to the sharing set obtained as the union of the sharing sets represented by each of the star sets. I.e., the sharing represented by $sss \in \wp(SS)$ is $\uplus sss = \cup\{ss^* \mid ss \in sss\}$. We will make use of these operations over sets of star sets and their following properties:

Lemma 6 *Let $sss \in \wp(SS)$ and $t \in Term$,*

$$\overset{*}{\cup} sss|_t = (\overset{*}{\cup} sss)_t \tag{17}$$

$$\overset{*}{\cup} \overline{sss|_t} = \overline{(\overset{*}{\cup} sss)_t} \tag{18}$$

Proof Using (5) and (10) we have that:

$$\overset{*}{\cup} sss|_t = \cup\{(ss|_t)^* \mid ss \in sss\} = \cup\{(ss^*)_t \mid ss \in sss\} = (\overset{*}{\cup} sss)_t$$

Using (6) and (11) we have that:

$$\overset{*}{\cup} \overline{sss|_t} = \cup\{\overline{ss|_t}^* \mid ss \in sss\} = \cup\{\overline{(ss^*)_t} \mid ss \in sss\} = \overline{(\overset{*}{\cup} sss)_t}$$

The Star-Sharing domain is

$$S^3H = \{(sss, sh) \mid sss \in \wp(SS), sh \in SH\}$$

and abstract unification in the domain is given, for equation $x = t$, where x is a variable and t a term, and $(sss, sh) \in S^3H$, by:

$$amgu^\omega(x = t, (sss, sh)) = (\overline{sss|_{xt}} \cup \{(\cup sss|_x \cup sh_x) \bowtie (\cup sss|_t \cup sh_t)\}, \overline{sh_{xt}})$$

Note that the sharing that an element $(sss, sh) \in S^3H$ represents corresponds to the sharing set obtained as the union of the sharing represented by sss and that represented by sh. I.e., the sharing represented by (sss, sh) is $\overset{*}{\cup} sss \cup sh$.

Theorem 3 *Let $(sss, sh) \in S^3H$ and equation $x = t$, where $x \in V$ and $t \in Term$, and $amgu^\omega(x = t, (sss, sh)) = (sss_\omega, sh_\omega)$,*

$$\overset{*}{\cup} sss_\omega \cup sh_\omega = amgu(x = t, \overset{*}{\cup} sss \cup sh) \tag{19}$$

Proof First note that:

$$\overset{*}{\cup} \overline{sss|_{xt}} \cup \overline{sh_{xt}} = \overline{(\overset{*}{\cup} sss \cup sh)_{xt}} \tag{20}$$

since $\overset{*}{\cup} \overline{sss|_{xt}} \cup \overline{sh_{xt}} = \overline{(\overset{*}{\cup} sss)_{xt}} \cup \overline{sh_{xt}} = \overline{(\overset{*}{\cup} sss \cup sh)_{xt}}$ by (18) and (11), respectively.

Also, note that:

$$(\cup sss|_t \cup sh_t)^* = (\overset{*}{\cup} sss \cup sh)^*_t \tag{21}$$

since $(\cup sss|_t \cup sh_t)^* = (\uplus sss|_t \cup sh_t)^* = ((\uplus sss)_t \cup sh_t)^* = (\uplus sss \cup sh)^*_t$ by (13), (17), and (10), respectively.

Now we can write:

$$
\begin{aligned}
& \uplus sss_\omega \cup sh_\omega \\
= \quad & \uplus(\overline{sss|_{xt}} \cup \{(\cup sss|_x \cup sh_x) \bowtie (\cup sss|_t \cup sh_t)\}) \cup \overline{sh_{xt}} \\
= \quad & \uplus\overline{sss|_{xt}} \cup ((\cup sss|_x \cup sh_x) \bowtie (\cup sss|_t \cup sh_t))^* \cup \overline{sh_{xt}} \\
= \quad & \uplus\overline{sss|_{xt}} \cup \overline{sh_{xt}} \cup ((\cup sss|_x \cup sh_x) \bowtie (\cup sss|_t \cup sh_t))^* \\
= \quad & \overline{(\uplus sss \cup sh)_{xt}} \cup ((\cup sss|_x \cup sh_x) \bowtie (\cup sss|_t \cup sh_t))^* && \text{by (20)} \\
= \quad & \overline{(\uplus sss \cup sh)_{xt}} \cup ((\cup sss|_x \cup sh_x)^* \bowtie (\cup sss|_t \cup sh_t)^*) && \text{by (2)} \\
= \quad & \overline{(\uplus sss \cup sh)_{xt}} \cup ((\uplus sss \cup sh)^*_x \bowtie (\uplus sss \cup sh)^*_t) && \text{by (21)} \\
= \quad & amgu(x = t, \uplus sss \cup sh)
\end{aligned}
$$

Thus, analysis with the Star-Sharing domain is as precise as analysis with the original Sharing domain. However, because of the absence of the costly star union operation in abstract unification, it is expected to be more efficient.

References

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A Sets of Star Sets

We can define abstract unification for sets of star sets as follows. For equation $x = t$, where x is a variable and t a term, and $sss \in \wp(SS)$:

$$amgu^{ss}(x = t, sss) = \overline{sss|_{xt}} \cup (sss|_x \uplus sss|_t)$$

and it is also correct and precise:

Theorem 4 *Let $sss \in \wp(SS)$ and equation $x = t$, $x \in V$, $t \in Term$,*

$$\uplus\, amgu^{ss}(x = t, sss) = amgu(x = t, \uplus\, sss) \tag{22}$$

Proof Under construction...

This was thought to remedy the probable loss of precision of $amgu^s$ by simply accumulating star sets in a set of them instead of merging them with set union. However, it seems that amg^s is not only correct but also precise, so $amgu^{ss}$ is not necessary.

The following is an alternative proof of (16).

Lemma 7

$$(ss|_t \cup sh_t)^* \supseteq (ss^* \cup sh)^*_t \tag{23}$$

Proof To prove this, let $s \in (ss^* \cup sh)^*_t$. From (10), $(ss^* \cup sh)^*_t = ((ss^*)_t \cup sh_t)^*$, so $s \in ((ss^*)_t \cup sh_t)^*$. Then there are $\{a_1, \ldots, a_m\} \subseteq (ss^*)_t$, $m \geq 0$, and $\{b_1, \ldots, b_n\} \subseteq sh_t$, $n \geq 0$, such that $s = (\cup_{i=1}^m a_i) \cup (\cup_{i=1}^n b_i)$, but not $m = n = 0$.

If $m = 0$ then $s = (\cup_{i=1}^n b_i)$, $n \geq 1$, so that $s \in sh_t^* \subseteq ((ss|_t)^* \cup sh_t^*) \subseteq (ss|_t \cup sh_t)^*$. So, let $m \geq 1$, whether $n = 0$ or $n \geq 1$.

For all $i = 1, \ldots, m$: $a_i \in ss^*$ and $a_i \cap t \neq \emptyset$; then there are $\{c_{i1}, \ldots, c_{ip}\} \subseteq ss$, $p \geq 1$, and $\{d_{i1}, \ldots, d_{iq}\} \subseteq ss$, $q \geq 0$, such that for all $j = 1, \ldots, p$, $c_{ij} \cap t \neq \emptyset$, for all $k = 1, \ldots, q$, $d_{ik} \cap t = \emptyset$, and $a_i = (\cup_{l=1}^p c_{il}) \cup (\cup_{l=1}^q d_{il})$. Then $c_{ij} \in ss_t$, $j = 1, \ldots, p$, and $d_{ik} \in \overline{ss_t}$, $k = 1, \ldots, q$, if they exist.

For any given $i \in \{1, \ldots, m\}$, all $j = 1, \ldots, p$, and $k = 1, \ldots, q$: If $q \geq 1$ then $(c_{ij} \cup d_{ik}) \in (ss_t \bowtie \overline{ss_t}) \subseteq ss|_t$; if $q = 0$ then we can write $c_{ij} \cup d_{ik} = c_{ij} \in ss_t \subseteq ss|_t$. Let, then, without loss of generality, $(c_{ij} \cup d_{ik}) \in ss|_t$. Thus, $(c_{ij} \cup d_{ik}) \in (ss|_t \cup sh_t)$.

Since $b_i \in sh_t$, for all $i = 1, \ldots, n$, then also $b_i \in (ss|_t \cup sh_t)$. Thus, $(\cup_{i=1}^m \cup_{j=1}^p \,{}_{k=1}^{q}(c_{ij} \cup d_{ik})) \cup (\cup_{i=1}^n b_i) = s \in (ss|_t \cup sh_t)^*$.

Hence, whether $m = 0$ or $m \geq 1$, $s \in (ss|_t \cup sh_t)^*$.