Automatic Inference of Resource Consumption Bounds

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The aim of COST ANALYSIS is to bound the resource consumption (aka cost) of executing a given program P on a given input data static

The aim of COST ANALYSIS is to bound the resource consumption (aka cost) of executing a given program P on a given input data *without actually executing* P

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- Upper Bounds (***worst case***)
- Lower Bounds (***best case***)

automatic

static

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- Upper Bounds (*worst case*)
- Lower Bounds (*best case*)
- Non-Asymptotic: $P(x) = 2 + 3{\cdot}x + 2{\cdot}x^2$
- Asymptotic: $P(x) = O(x^2)$

WHAT IS COST ANALYSIS?

automatic

static

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- Execution steps
- Visits to specific program points
- Memory (possibly with garbage collection)

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- Time? Energy?

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  noncumulative
- ~~Time? Energy?~~
  platform dependent

State of the Art

- Work on automatic cost analysis dates back to 1975, with the seminal work of Wegbreit
- His system was able to compute:
  - interesting results, but for
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- Since then, a number of analyses and systems have been built which extend the capabilities of cost analysis:
  - functional programs [Le Metayer'88, Rosendahl'89, Wadler'88, Sands'95, Benzinger'04, ..., Hofmann'10, ...]
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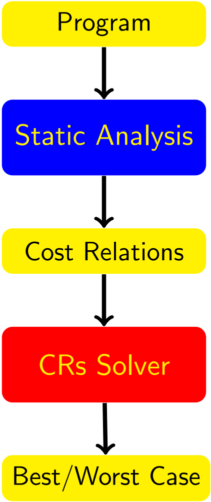
ESOP'07

A Classical Approach to Cost Analysis



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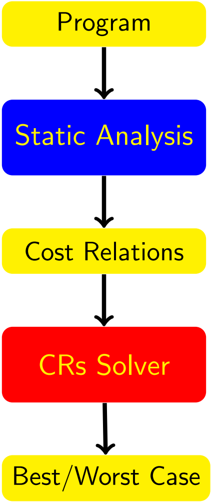
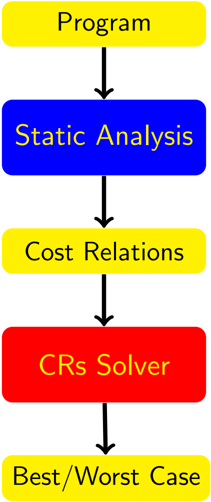


A classical approach [Wegbreit'75] to cost analysis consists of:

1. expressing the cost of a program part in terms of other program parts, thus obtaining *recurrence relations*
2. solving the relations by obtaining a *closed-form* for the cost in terms of the input arguments



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The current situation is that

- Most work has concentrated on the 1^{st} phase
- difficulties of the 2^{nd} phase have been overseen
- usage of cost analysis requires both!
- COSTA we address both phases.

What do we expect from cost analysis?

```
static void sort(int a[]) {

   for (int i=a.length-2; i>=0; i--) {

q     int j=i+1;
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Worst-Case (UB)

$$sort(a) = q * (a-1) + p * (a-1)^2$$

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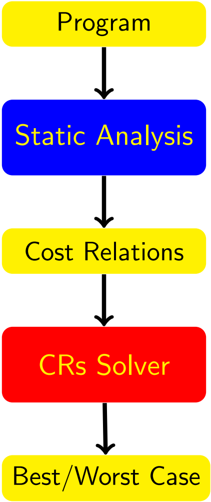
Worst-Case (UB)

$sort(a) = q * (a-1) + p * (a-1)^2$

Best-Case (LB)

$sort(a) = q * (a-1)$

First phase: Generation of Cost Relations



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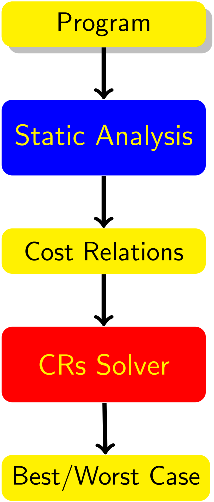


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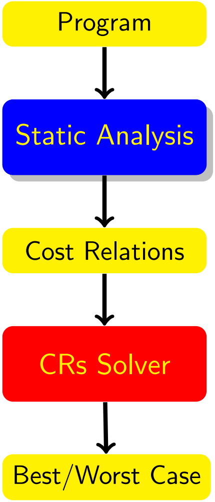


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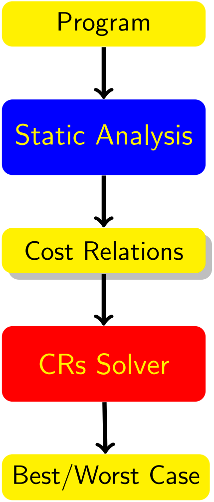


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$$\begin{array}{ll}
sort(a) = B(a,i) & \{i=a-2,\ a\geq 0\} \\
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B(a,i) = q + C(a,j) + B(a,i') & \{i\geq 0,\ j=i+1,\ i'=i-1\} \\
C(a,j) = 0 & \{j\geq a\} \\
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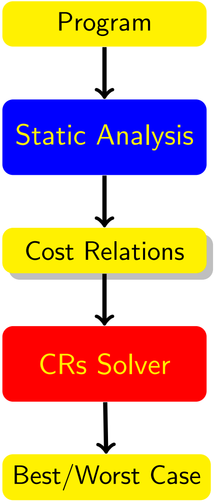


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Extracting loops

Loops are extracted from the CFG of the program and cost relations are generated loop by loop

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Cost Relations

CRs Solver

Best/Worst Case

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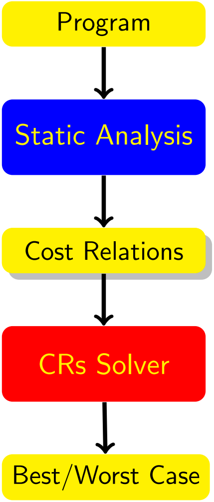


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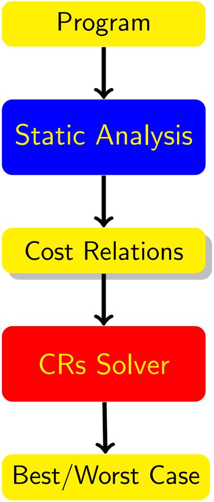


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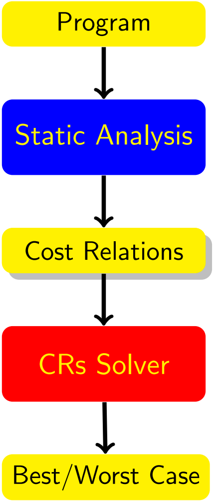


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Size analysis

Global size analysis is performed in order to infer how the sizes of data change along execution

Cost Relations

CRs Solver

Best/Worst Case

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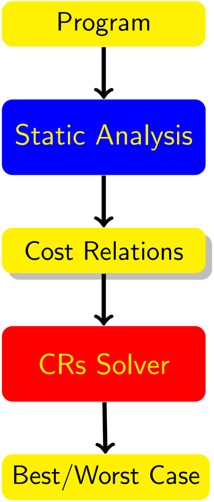


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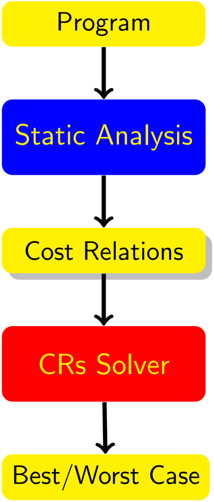


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First phase: Generation of Cost Relations

Program

Static Analysis

Cost Relations

CRs Solver

Best/Worst Case

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Modular definition

Cost relations define the cost of executing a program fragment in terms of the cost of executing other fragments

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CRs Solver

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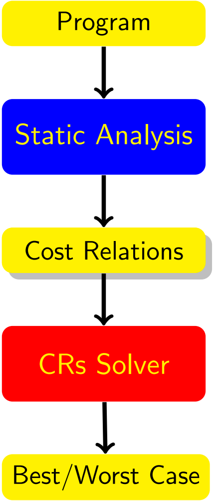
First phase: Generation of Cost Relations

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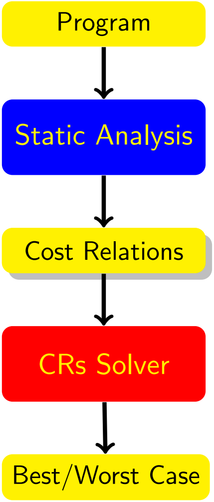
First phase: Generation of Cost Relations

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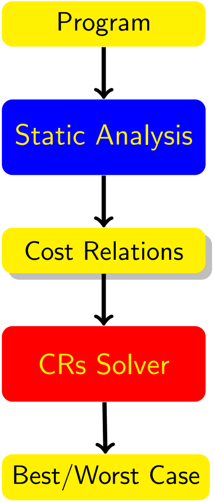


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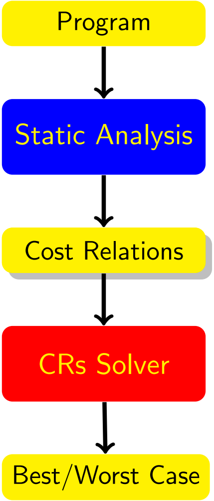


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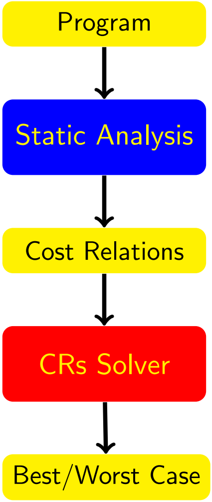


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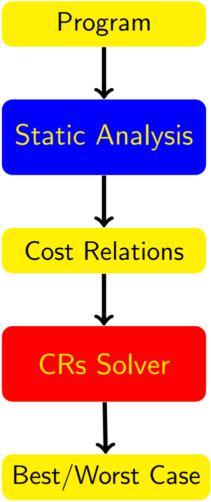


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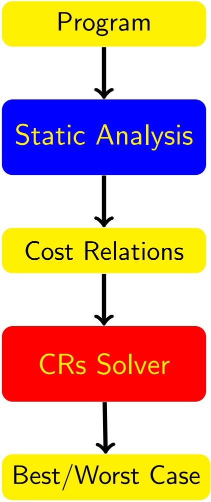


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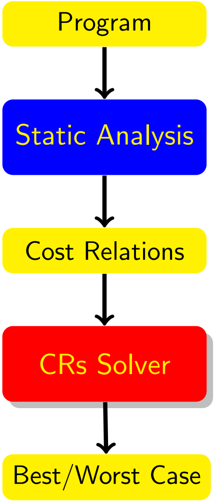


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Program ⟶ **Static Analysis** ⟶ Cost Relations

- Apply class analysis to compute the reachable code

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Program $\longrightarrow$ Static Analysis $\longrightarrow$ Cost Relations

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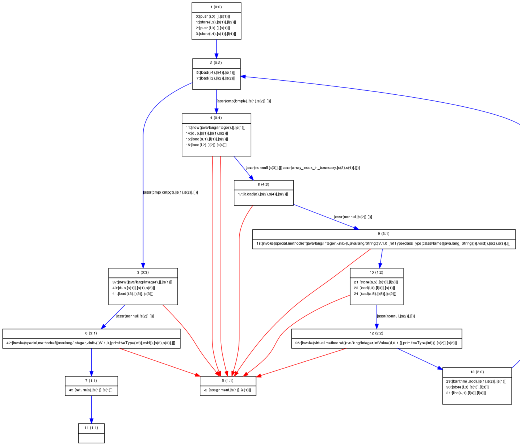
- Ap … e
- Co



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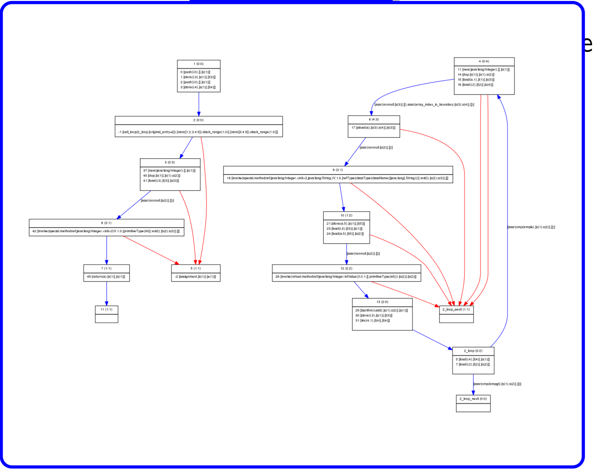

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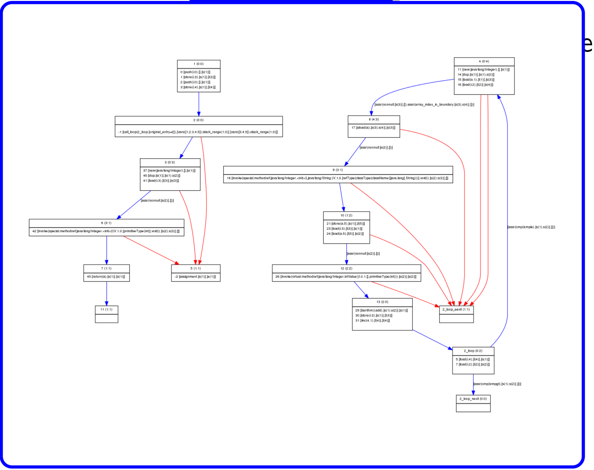

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```
while ( x != null ) {
  x = x.next;
}
```

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C(a,j)=0                {j>=a}
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- References: ESOP'07, TCS'12

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CAS can obtain an exact closed-form solution for:

$$P(0) = 0$$
$$P(n) = E + P(n-1) + \cdots + P(n-1)$$

deterministic, 1 base-case, 1 recursive case, 1 argument

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- Wh
- CRs

Several possible runs for $C(10,1)$

$C(10,1) \rightarrow C(10,2)$
$C(10,1) \rightarrow C(10,2) \rightarrow C(10,3)$
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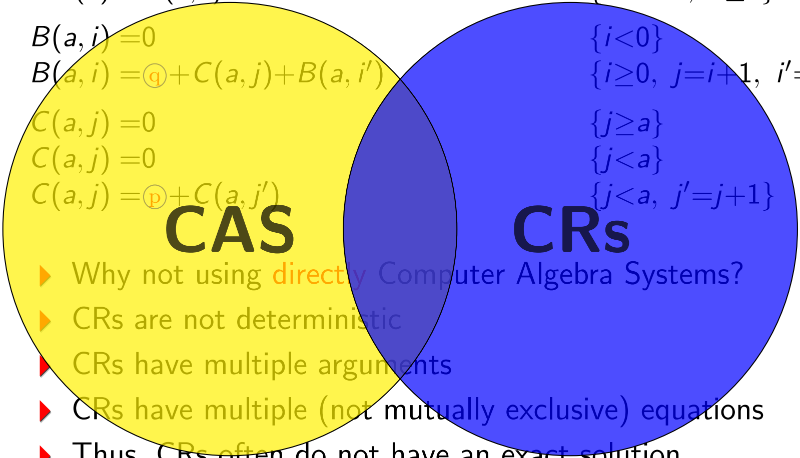

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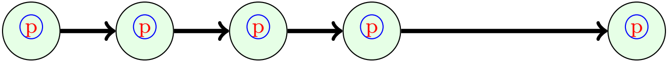
- An evaluation for $C(a_0, j_0)$ looks like:



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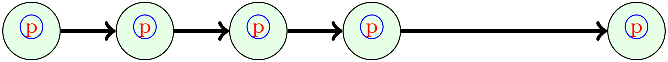

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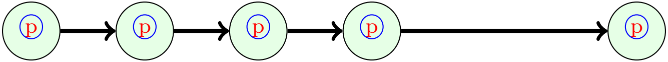

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A Practical Upper Bound Solver (PUBS)

Ranking function

We are seeking a (ranking) function $\hat{f}$ that maps the program states to integers, such that

$$\forall i,j,j'.\ \varphi \models \hat{f}(i,j) - \hat{f}(i,j') \geq 1 \ \wedge\ \hat{f}(i,j) \geq 0$$

There are automatic techniques for synthesizing such functions [Sohn and Van Gelder 1991, Podelski and Rybalchenko 2004]

so … $0\}$

$B($ …

$B($ … $,\ i'{=}i{-}1\}$

$C($ …

$C($ …

$C($ … $\}$

- A …

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$$
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sort(a) = B(a,i) & \{i{=}a{-}2,\ a{\geq}0\} \\
B(a,i) = 0 & \{i{<}0\} \\
B(a,i) = q + C(a,j) + B(a,i') & \{i{\geq}0,\ j{=}i{+}1,\ i'{=}i{-}1\} \\
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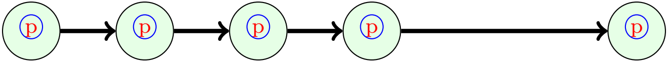
\end{array}
$$

- An evaluation for $C(a_0, j_0)$ looks like:



- How many p has this chain ?

$$\hat{f}(a_0, j_0) = \mathtt{nat}(a_0 - j_0)$$

A Practical Upper Bound Solver (PUBS)

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$$C(a_0, j_0) = p * \mathtt{nat}(a_0 - j_0)$$

- An evaluation for $C(a_0, j_0)$ looks like:

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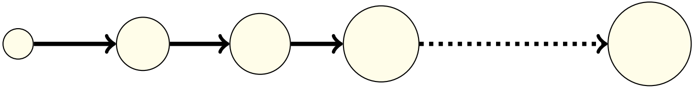

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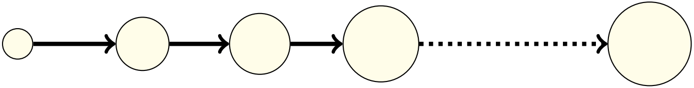

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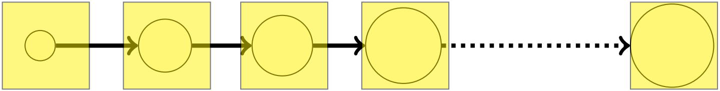

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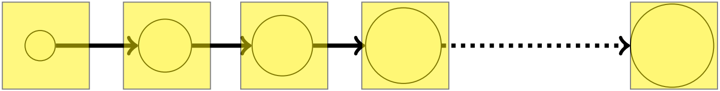

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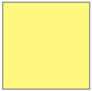
- Worst-case is  $*\ \mathtt{nat}(i_0+1)$

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Inferring the box (or maximizing expression)

- What is the maximum value that $q + p * \mathtt{nat}(a - j)$ can take in terms of the initial values $\langle a_0, i_0 \rangle$?

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- Infer an invariant $\langle B(a_0, i_0) \rightsquigarrow B(a, i), \Psi \rangle$
- Use (parametric) integer programming to maximize $a - j$ w.r.t $\Psi \wedge \varphi$ and the parameters $\langle a_0, i_0 \rangle$

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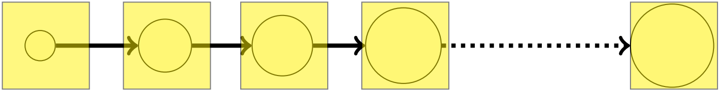

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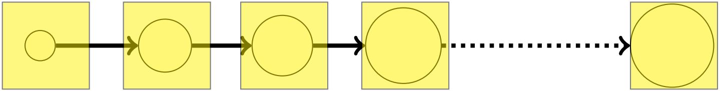

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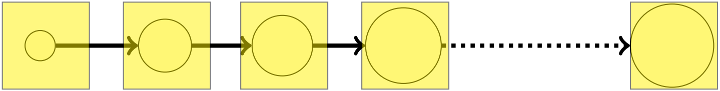

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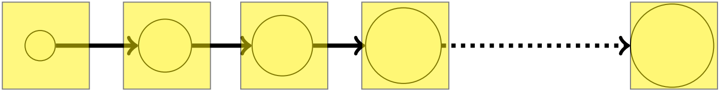

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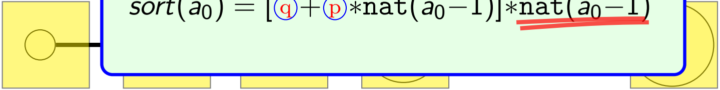

Worst-case UB for *sort*

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- References: SAS'08, JAR'11, VMCAI'11

Theorem (soundness)

- *Let $P(\bar{x})$ be a method,*
- *R the resource we are measuring,*
- *$UB(\bar{x})$ the upper bound computed from P.*

For any valid input $\bar{v}$, if there exists a trace t from $P(\bar{v})$, then we ensure $\boxed{UB(\bar{v}) \geq R(t)}$

Advanced Topics in Current Research

- Non-cumulative types of resources (memory)
- Certification of results
- Handling the shared memory (heap)
- Modularity, incrementality of analysis
- Concurrency in cost analysis
- Implementing a cost analyzer

Non-Cumulative Resources

- Total memory consumption (ignoring GC):
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    - "When" it is applied (scope-based vs. ideal)
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- Assumption: scope-based GC based on reachability
  - Scoped memory managers are common.
  - Useful for estimating consumption in systems with stack-allocation.

Memory Consumption with Scope-based GC

```
class Tree {
  int data;
  Tree left;
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  void print() {
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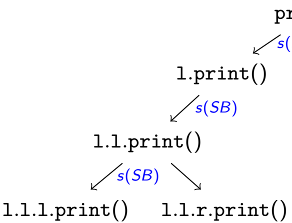

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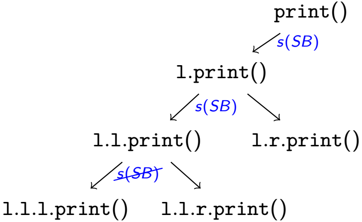

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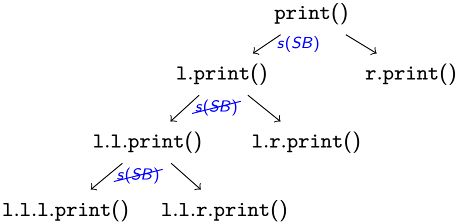

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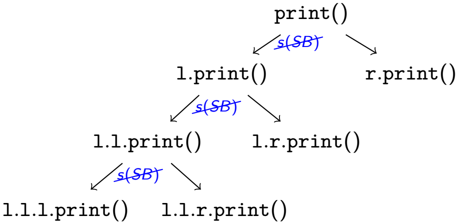
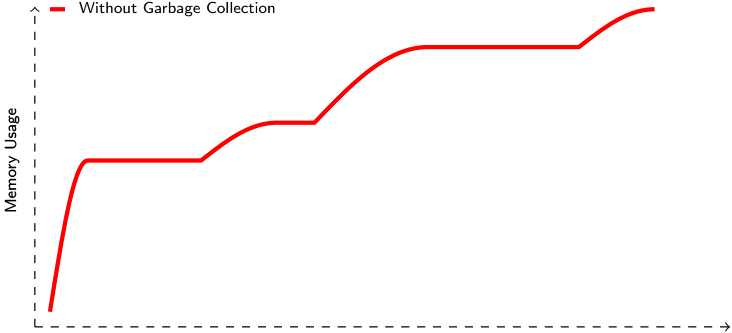


Cost Analysis - Peak Resource Consumptions

COST ANALYSIS - PEAK RESOURCE CONSUMPTIONS



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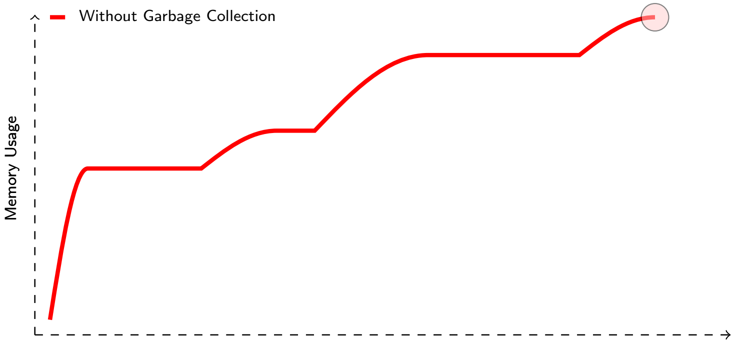

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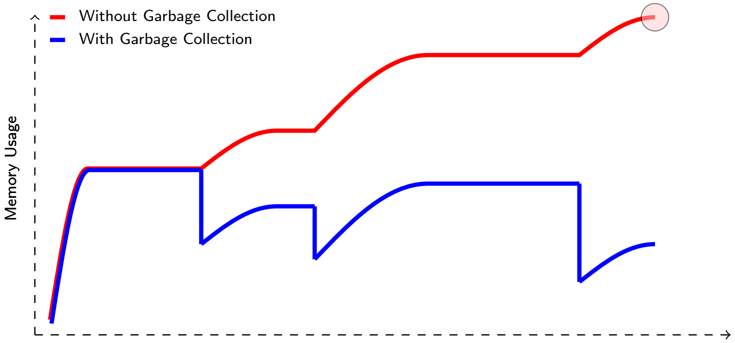

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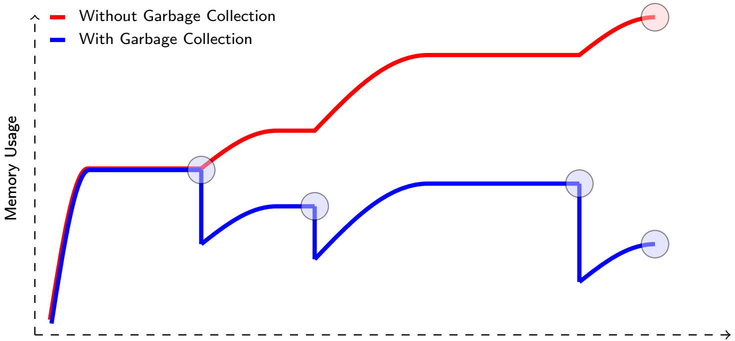

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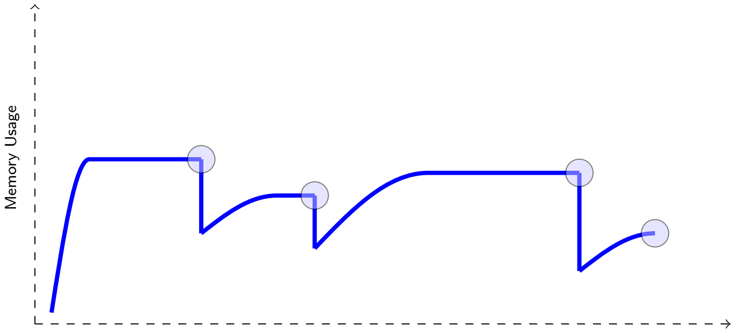

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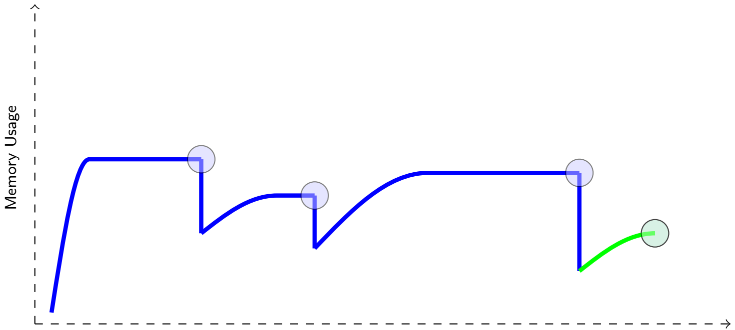

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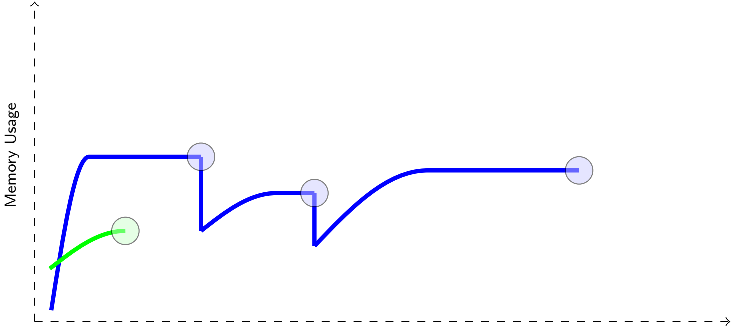

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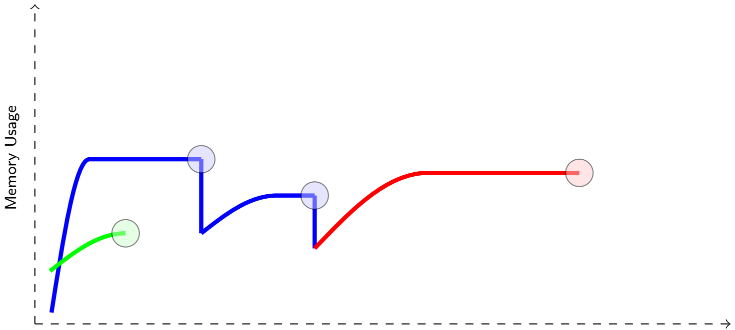

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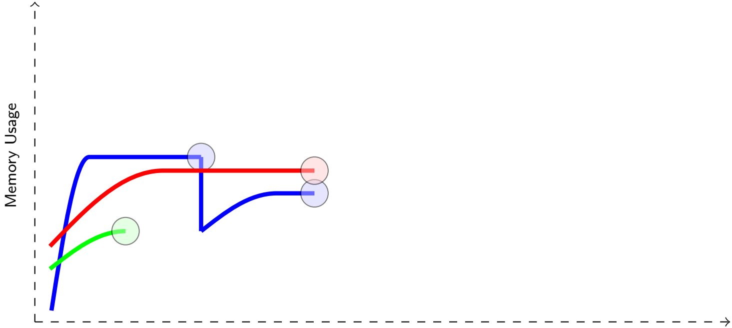
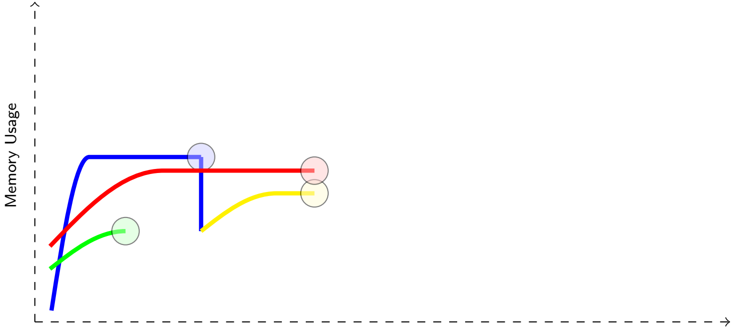
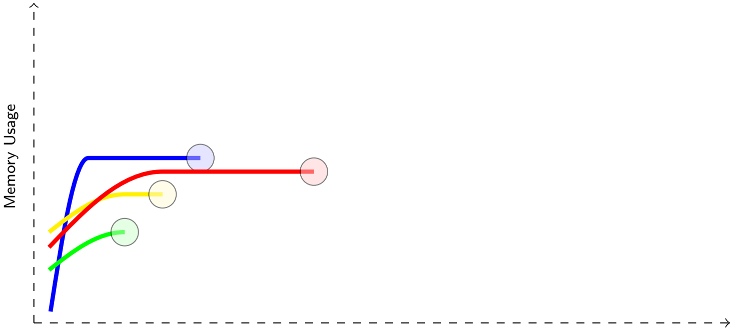
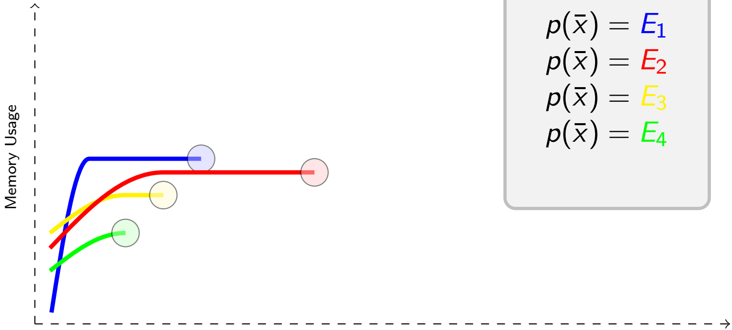


Cost Analysis - Peak Resource Consumptions




Memory Usage



$$p(\bar{x}) = E_1$$
$$p(\bar{x}) = E_2$$
$$p(\bar{x}) = E_3$$
$$p(\bar{x}) = E_4$$

Example (Scope-based GC)

```
class Tree {
  int data;
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Total Consumption Equations:

$$\mathtt{print}(t) = \mathrm{s(SB)} \qquad \{t = 1\}$$
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EXAMPLE (SCOPE-BASED GC)

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TOTAL CONSUMPTION EQUATIONS:

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UPPER BOUND ON TOTAL: $\texttt{print}^{\texttt{ub}}(t) = 2^t * \mathrm{s}(\mathrm{SB})$

UPPER BOUND ON PEAK:

$\hat{\texttt{print}}^{\texttt{ub}}(t) = t * \mathrm{s}(\mathrm{SB})$

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Example (Scope-based GC)

Total Consumption Equations:

```
cl
```

$,t > t''\}$

$s(SB)$

- What does it mean that *e* is an upper bound on the *peak* heap consumption?

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Upper Bound on Peak:

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```
System.out.println("data="+data);
```

1 // new StringBuffer(...)

Upper Bound on Peak:

2

B)

1. If we set the memory limit to e; *and*
2. Assume that GC will always collect unreachable objects upon method's return; *then*
3. The program will execute without running out of memory.

}

P

pr

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- Possible approaches:
  - Perform full-blown verification of COSTA
  - Formally verify the results obtained by COSTA
- Selected alternative: construct a validating tool which, after every run of COSTA, formally confirms results are correct
  - COSTA $\Rightarrow$ Generates the upper bounds
  - KeY $\Rightarrow$ Verifies the correctness of the upper bounds

COSTA and KeY

- COSTA: Inferring Upper Bounds
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  - Java Card Dynamic Logic, and Symbolic Execution
  - JML is used for specifications
- COSTA +KeY:
  - COSTA outputs JML annotations on the Java source
  - KeY reads annotated Java source code to verify the correctness of all JML annotations, and generates a formal proof
  - The process is fully automatic

EXAMPLE

```
int f(int n) {
  int x=0;



  while (n>0) {
    x += n;
    n--;
  }
  return x;
}
```

- COSTA infers the Upper Bound $f(n) = 7 + 7 * \mathtt{nat}(n)$

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- KeY is used to verify the annotated program

TOOL DEMO

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- Future Directions
  - handle Java multithreaded programs
  - handle cyclic data structures
  - average case, and distribution

Credits - the COSTA team

http://costa.ls.fi.upm.es

Elvira Albert
Diego Alonso
Puri Arenas
Jesús Correas
Antonio Flores
Samir Genaim
Miguel Gómez-Zamalloa
Abu Naser Masud
Jose Miguel Rojas
Germán Puebla
Diana Ramírez
Guillermo Román
Damiano Zanardini