Computational Logic

CLP Semantics and Fundamental Results

Constraint Domains

- Semantics parameterized by the constraint domain:
  CLP($\mathcal{X}$), where $\mathcal{X} \equiv (\Sigma, \mathcal{D}, \mathcal{L}, \mathcal{T})$
- Signature Σ: set of predicate and function symbols, together with their arity
- $\mathcal{L} \subseteq \Sigma$–formulae: constraints
- D is the set of actual elements in the domain
- Σ–structure $\mathcal{D}$: gives the meaning of predicate and function symbols (and hence, constraints).
- $\mathcal{T}$ a first–order theory (axiomatizes some properties of $\mathcal{D}$)
- $(\mathcal{D}, \mathcal{L})$ is a *constraint domain*
- Assumptions:
  - ◇ $\mathcal{L}$ built upon a first–order language
  - ◇ $= \in \Sigma$ is identity in $\mathcal{D}$
  - ◇ There are identically false and identically true constraints in $\mathcal{L}$
  - ◇ $\mathcal{L}$ is closed w.r.t. renaming, conjunction and existential quantification

Domains (I)

- $\Sigma = \{0, 1, +, *, =, <, \leq\}$, D = **R**, $\mathcal{D}$ interprets Σ as usual, $\Re = (\mathcal{D}, \mathcal{L})$
  - ◇ Arithmetic over the reals
  - ◇ Eg.: $x^2 + 2xy < \frac{y}{x} \wedge x > 0 \;\; (\equiv xxx + xxy + xxy < y \wedge 0 < x)$
- Question: is 0 needed? How can it be represented?

- Let us assume $\Sigma' = \{0, 1, +, =, <, \leq\}$, $\Re_{Lin} = (\mathcal{D}', \mathcal{L}')$
  - ◇ Linear arithmetic
  - ◇ Eg.: $3x - y < 3 \;\; (\equiv x + x + x < 1 + 1 + 1 + y)$

- Let us assume $\Sigma'' = \{0, 1, +, =\}$, $\Re_{LinEq} = (\mathcal{D}'', \mathcal{L}'')$
  - ◇ Linear equations
  - ◇ Eg.: $3x + y = 5 \wedge y = 2x$

Domains (II)

- $\Sigma = \{<$*constant and function symbols*$>, =\}$
- D = { finite trees }
- $\mathcal{D}$ interprets Σ as tree constructors
- Each $f \in \Sigma$ with arity n maps n trees to a tree with root labeled f and whose subtrees are the arguments of the mapping
- Constraints: syntactic tree equality
- $\mathcal{FT} = (\mathcal{D}, \mathcal{L})$
  - ◇ Constraints over the Herbrand domain
  - ◇ Eg.: $g(h(Z), Y) = g(Y, h(a))$
- LP $\equiv$ CLP($\mathcal{FT}$)

Domains (III)

- $\Sigma = \{<$*constants*$>, \lambda, ., ::, =\}$
- D = $\{$ finite strings of constants $\}$
- $\mathcal{D}$ interprets . as string concatenation, :: as string length
  - Equations over strings of constants
  - Eg.: $X.A.X = X.A$

- $\Sigma = \{0, 1, \neg, \wedge, =\}$
- D = $\{true, false\}$
- $\mathcal{D}$ interprets symbols in Σ as boolean functions
- $\mathcal{BOOL} = (\mathcal{D}, \mathcal{L})$
  - Boolean constraints
  - Eg.: $\neg(x \wedge y) = 1$

CLP($\mathcal{X}$) Programs

- Recall that:
  - ◇ Σ is a set of predicate and function symbols
  - ◇ $\mathcal{L} \subseteq \Sigma$–formulae are the constraints
- Π: set of predicate symbols definable by a program
- Atom: $p(t_1, t_2, \ldots, t_n)$, where $t_1, t_2, \ldots, t_n$ are terms and $p \in \Pi$
- Primitive constraint: $p(t_1, t_2, \ldots, t_n)$, where $t_1, t_2, \ldots, t_n$ are terms and $p \in \Sigma$ is a predicate symbol
- Every constraint is a (first–order) formula built from primitive constraints
- The class of constraints will vary (generally only a subset of formulas are considered constraints)
- A CLP program is a collection of rules of the form $a \leftarrow b_1, \ldots, b_n$ where a is an atom and the b_i's are atoms or constraints
- A fact is a rule $a \leftarrow c$ where c is a constraint
- A goal (or query) G is a conjunction of constraints and atoms

Basic Operations on Constraints

- Constraint domains are expected to support some basic operations on constraints
  1. Consistency (or satisfiability) test: $\mathcal{D} \models \tilde{\exists} c$,
  2. Implication or entailment: $\mathcal{D} \models c_0 \rightarrow c_1$,
  3. Projection of a constraint c_0 onto variables $\tilde{x}$ to obtain a constraint c_1 such that $\mathcal{D} \models c_1 \leftrightarrow \exists_{-\tilde{x}} c_0$,
  4. Detection of uniqueness of variable value: $\mathcal{D} \models c(x, \tilde{z}) \wedge c(y, \tilde{w}) \rightarrow x = y$
- Actually, only the first one is really required
- In actual implementations, some of these operations—in particular the test of consistency—may be incomplete
- Examples:
  - ◇ $x * x < 0$ is inconsistent in $\Re$ (because $\neg \exists x \in \Re : x * x < 0$)
  - ◇ $\mathcal{D} \models (x \wedge y = 1) \rightarrow (x \vee y = 1)$ in $\mathcal{BOOL}$
  - ◇ In $\mathcal{FT}$, the projection of $x = f(y) \wedge y = f(z)$ on $\{x, z\}$ is $x = f(f(z))$
  - ◇ In $\mathcal{WE}$, $\mathcal{D} \models x.a.x = x.a \wedge y.b.y = y.b \rightarrow x = y$
- Prove the last assertion!

Properties of CLP Languages

- $\mathcal{T}$ axiomatizes some of the properties of $\mathcal{D}$
- For a given Σ, let $(\mathcal{D}, \mathcal{L})$ be a constraint domain with signature Σ, and $\mathcal{T}$ a Σ–theory.
- $\mathcal{D}$ and $\mathcal{T}$ correspond on $\mathcal{L}$ if:
  - ◇ $\mathcal{D}$ is a model of $\mathcal{T}$, and
  - ◇ for every constraint $c \in \mathcal{L}$, $\mathcal{D} \models \tilde{\exists} c$ iff $\mathcal{T} \models \tilde{\exists} c$.
- $\mathcal{T}$ is *satisfaction complete* with respect to $\mathcal{L}$ if for every constraint $c \in \mathcal{L}$, either $\mathcal{T} \models \tilde{\exists} c$ or $\mathcal{T} \models \neg\tilde{\exists} c$.
- $(\mathcal{D}, \mathcal{L})$ is *solution compact* if

$$\forall c \exists \{c_i\}_{i \in I} : \mathcal{D} \models \forall \tilde{x} \neg c(\tilde{x}) \longleftrightarrow \bigvee_{i \in I} c_i(\tilde{x})$$

i.e., any negated constraint in $\mathcal{L}$ can be expressed as a (in)finite disjunction of constraints

Solution Compactness

- Important to lift SLDNF results to CLP($\mathcal{X}$)
- We have to deal only with user predicates
- E.g.
  - $x \not\geq y$ in CLP($\Re$) is $x < y$
  - $x \neq y$ in CLP($\Re$) is $x < y \vee y < x$
  - $\Re_{Lin}$ with constraint $x \neq \pi$ is not s.c.
- How can we express $x \neq y$ in CLP($\mathcal{FT}$)?

Logical Semantics (I)

- Two common logical semantics exist.
- The first one interprets a rule

$$p(\tilde{x}) \leftarrow b_1, \ldots, b_n$$

as the logic formula

$$\forall \tilde{x}, \tilde{y}\ p(\tilde{x}) \vee \neg b_1 \vee \ldots \vee \neg b_n$$

Logical Semantics (II)

- The second one associates a logic formula to each predicate in Π
  - ◇ If the set of rules of P with p in the head is:

$$\begin{array}{rcl} p(\tilde{x}) & \leftarrow & B_1 \\ p(\tilde{x}) & \leftarrow & B_2 \\ & \vdots & \\ p(\tilde{x}) & \leftarrow & B_n \end{array}$$

    then the formula associated with p is:

$$\begin{array}{rcl} \forall \tilde{x}\ p(\tilde{x}) & \leftrightarrow & \exists \tilde{y}_1 B_1 \\ & \vee & \exists \tilde{y}_2 B_2 \\ & \vdots & \\ & \vee & \exists \tilde{y}_n B_n \end{array}$$

  - ◇ If p does not occur in the head of a rule of P, the formula is: $\forall \tilde{x} \neg p(\tilde{x})$
  - ◇ The collection of all such formulas is the *Clark completion* of P (denoted by P^*)
- These two semantics differ on the treatment of the negation

Logical Semantics (III)

- A *valuation* is a mapping from variables to D, and the natural extension which maps terms to D and formulas to closed $\mathcal{L}^*$–formulas.
- A $\mathcal{D}$–interpretation of a formula is an interpretation of the formula with the same domain as $\mathcal{D}$ and the same interpretation for the symbols in Σ as $\mathcal{D}$.
- It can be represented as a subset of $B_{\mathcal{D}}$ where

$$B_{\mathcal{D}} = \{p(\tilde{d}) \mid p \in \Pi, \tilde{d} \in D^k\}$$

- A $\mathcal{D}$–model of a closed formula is a $\mathcal{D}$–interpretation which is a model of the formula.
- The usual logical semantics is based on the $\mathcal{D}$–models of P and the models of P^*, $\mathcal{T}$.
- The least $\mathcal{D}$–model of a formula Q is denoted by $lm(Q, \mathcal{D})$.
- A *solution* to a query G is a valuation v such that $v(G) \subseteq lm(P, \mathcal{D})$.

Fixpoint Semantics

- Based on one-step consequence operator $T_P^{\mathcal{D}}$ (also called "immediate consequence operator").
- Take as semantics $lfp(T_P^{\mathcal{D}})$, where:

$$T_P^{\mathcal{D}}(I) = \{p(\tilde{d}) \mid p(\tilde{x}) \leftarrow c, b_1, \ldots, b_n \in P, a_i \in I, \\ \mathcal{D} \models v(c), v(\tilde{x}) = \tilde{d}, v(b_i) = a_i\}$$

- Theorems:

1. $T_P^{\mathcal{D}} \uparrow \omega = lfp(T_P^{\mathcal{D}})$
2. $lm(P, \mathcal{D}) = lfp(T_P^{\mathcal{D}})$

Top–Down Operational Semantics (I)

- General framework for operational semantics
- Formalized as a transition system on *states*
- State: a 3–tuple $\langle A, C, S \rangle$, or $fail$, where
  - A is a multiset of atoms and constraints,
  - $C \cup S$ multiset of constraints,
  - C, active constraints (awake)
  - S, passive constraints (asleep)
- *Computation* and *Selection* rules depend on A
- Transition system: parameterized by a predicate $consistent$ and a function $infer$:
  - $consistent(C)$ checks the consistency of a constraint store
  - Usually "$consistent(C)$ iff $\mathcal{D} \models \tilde{\exists} c$", but sometimes "if $\mathcal{D} \models \tilde{\exists} c$ then $consistent(C)$"
  - $infer(C, S)$ computes a new set of active and passive constraints

Top–Down Operational Semantics (II)

- Transition r: computation step; rewriting using user predicates
  $\langle A \cup a, C, S\rangle \rightarrow_r \langle A \cup B, C, S \cup (a = h)\rangle$
  if $h \leftarrow B \in P$, and a and h have the same predicate symbol, or
  $\langle A \cup a, C, S\rangle \rightarrow_r fail$
  if there is no rule $h \leftarrow B$ of P such that a and h have the same predicate symbol
  ($a = h$ is a set of argument–wise equations) if a is a predicate symbol selected by the computation rule

- Transition c: selects constraints
  $\langle A \cup c, C, S\rangle \rightarrow_c \langle A, C, S \cup c\rangle$
  if c is a constraint selected by the computation rule

- Transition i: infers new constraints
  $\langle A, C, S\rangle \rightarrow_i \langle A, C', S'\rangle$ if $(C', S') = infer(C, S)$
  - In particular, may turn passive constraints into active ones

- Transition s: checks satisfiability

$$\langle A, C, S\rangle \rightarrow_s \begin{cases} \langle A, C, S\rangle & \text{if } consistent(C) \\ fail & \text{if } \neg consistent(C) \end{cases}$$

Top–Down Operational Semantics (III)

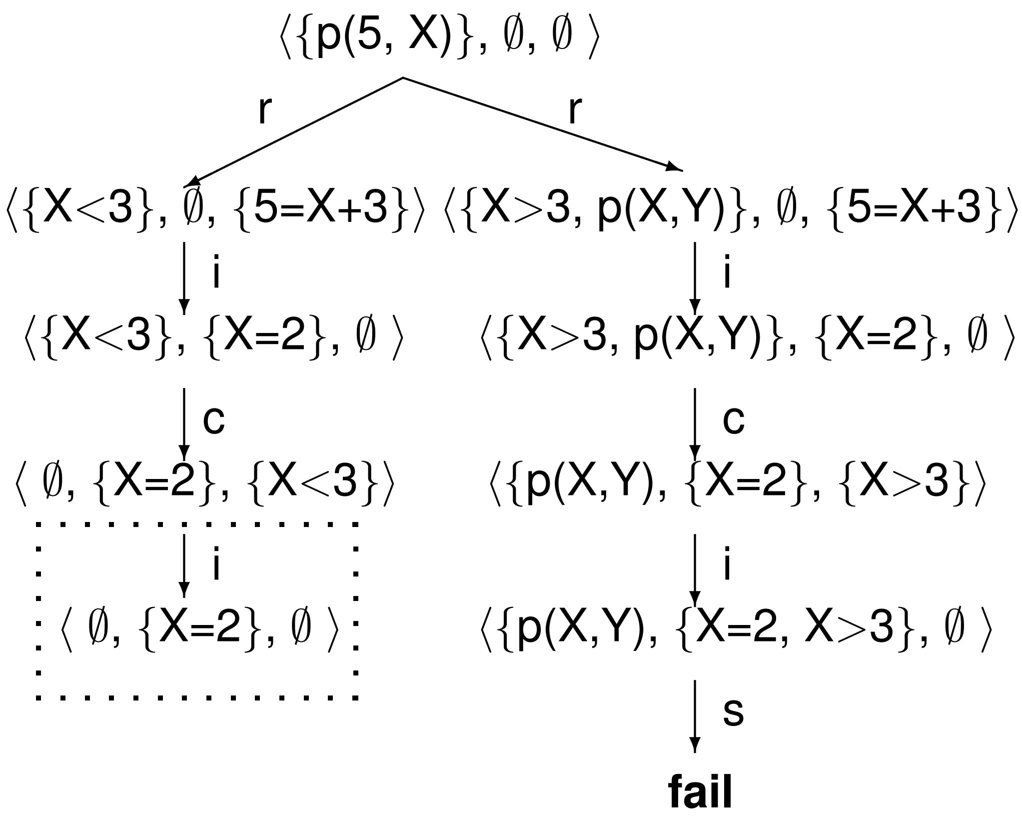
- Initial state: $\langle G, \emptyset, \emptyset \rangle$
- Derivation: $\langle A_1, C_1, S_1 \rangle \rightarrow \ldots \rightarrow \langle A_i, C_i, S_i \rangle \rightarrow \ldots$
- Final state: $E \rightarrow E$
- *Successful derivation*: final state $\langle \emptyset, C, S \rangle$
- A derivation *flounders* if finite and the final state is $\langle A, C, S \rangle$ with $A \neq \emptyset$
- A derivation is *failed* if it is finite and the final state is fail
- Answer: $\exists_{-\tilde{x}} C \wedge S$, where $\tilde{x}$ are the variables in the initial goal
- A derivation is *fair* if it is failed or, for every i and every $a \in A_i$, a is rewritten in a later transition
- A computation rule is fair if it gives rise only to fair derivations

Top–Down Operational Semantics (IV)

- *Computation tree* for goal G and program P:
  - Nodes labeled with states
  - Edges labeled with $\rightarrow_r$, $\rightarrow_c$, $\rightarrow_i$ or $\rightarrow_s$
  - Root labeled by $\langle G, \emptyset, \emptyset \rangle$
  - All sons of a given node have the same label
  - Only one son with transitions $\rightarrow_c$, $\rightarrow_i$ or $\rightarrow_s$
  - A son per program clause with transition $\rightarrow_r$

Computation Tree: Example

- Consider the program
  $p(X+3, X) \leftarrow X < 3.$
  $p(X+3, X) \leftarrow X > 3, p(X, Y).$
  and the goal $\leftarrow p(5, X)$
- A possible computation tree is:



- Dotted rectangle: previous state was final as well

Types of CLP($\mathcal{X}$) Systems

- *Quick–checking* CLP($\mathcal{X}$) system: its operational semantics can be described by $\rightarrow_{ris} \equiv \rightarrow_r \rightarrow_i \rightarrow_s$ and $\rightarrow_{cis} \equiv \rightarrow_c \rightarrow_i \rightarrow_s$
- I.e., always selects either an atom or a constraint, infers and checks consistency
- *Progressive* CLP system: for all $\langle A, C, S \rangle$ with $A \neq \emptyset$, every derivation from that state either fails or contains a $\rightarrow_r$ or $\rightarrow_c$ transition
- *Ideal* CLP system:
  - Quick-checking
  - Progressive
  - $infer(C, S) = (C \cup S, \emptyset)$
  - $consistent(C)$ holds iff $\mathcal{D} \models \tilde{\exists} c$

Soundness and Completeness Results

- Success set: the set of queries plus constraints which have a successful derivation in the program:
  $SS(P) = \{p(\tilde{x}) \leftarrow c \mid \langle p(\tilde{x}), \emptyset, \emptyset\rangle \rightarrow^* \langle \emptyset, c', c''\rangle, \mathcal{D} \models c \leftrightarrow \exists_{-\tilde{x}} c' \wedge c''\}$
- Consider a program P in the CLP language determined by a 4–tuple $(\Sigma, \mathcal{D}, \mathcal{L}, \mathcal{T})$ and executing on an **ideal** CLP system. Then:

  1. $[SS(P)]_{\mathcal{D}} = lm(P, D)$, where
  $$[SS(P)]_{\mathcal{D}} = \{v(a) \mid (a \leftarrow c) \in SS(P), \mathcal{D} \models v(c)\}$$
  2. $SS(P) = lfp(S_P^{\mathcal{D}})$
  3. (Soundness) if the goal G has a successful derivation with answer constraint c, then $P, \mathcal{T} \models c \rightarrow G$
  4. (Completeness) if $P, \mathcal{T} \models c \rightarrow G$ then there are derivations for the goal G with answer constraints $c_1, \ldots, c_n$ such that $\mathcal{T} \models c \rightarrow \bigvee_{i=1}^{n} c_i$
  5. Assume $\mathcal{T}$ is satisfaction complete w.r.t. $\mathcal{L}$. Then the goal G is finitely failed for P iff $P^*, \mathcal{T} \models \neg G$.

Negation in CLP($\mathcal{X}$)

- Most LP results can be lifted to CLP($\mathcal{X}$)
- In particular, negation as failure (à la SLDNF) is still valid using:
  - ◇ Satisfiability instead of unification
  - ◇ Variable elimination instead of groundness
- Added bonus: if the system is *solution compact*, then negated constraints can be expressed in terms of primitive constraints
- Less chances of a floundered / incorrect computation