Computational Logic

Automated Deduction Fundamentals
Elements of First-Order Predicate Logic

First Order Language:

- An *alphabet* consists of the following classes of symbols:
  1. *variables* denoted by $X, Y, Z, Boo, ...$, (infinite)
  2. *constants* denoted by $1, a, boo, john, ...$
  3. *functors* denoted by $f, g, +, −, ..$
  4. *predicate symbols* denoted by $p, q, dog, ...$
  5. *connectives*, which are: $\neg$ (negation), $\lor$ (disjunction), $\land$ (conjunction), $\rightarrow$ (implication) and $\leftrightarrow$ (equivalence),
  6. *quantifiers*, which are: $\exists$ (there exists) and $\forall$ (for all),
  7. *parentheses*, which are: ( and ) and the *comma*, that is: “,”.

- Each functor and predicate symbol has a fixed *arity*, they are often represented in *Functor/Arity* form, e.g. f/3.
- A constant can be seen as a functor of arity 0.
- Propositions are represented by a predicate symbol of arity 0.
Important: Notation Convention Used

(A bit different from standard notational conventions in logic, but good for compatibility with LP systems)

- Variables: start with a capital letter or a “_” (X, Y, a, 1)
- Atoms, functors, predicate symbols: start with a lower case letter or are enclosed in ’ ’ (f, g, a, 1, x, y, z, ’X’, ’1’)
Terms and Atoms

We define by induction two classes of strings of symbols over a given alphabet.

- The class of *terms*:
  - a variable is a term,
  - a constant is a term,
  - if \( f \) is an \( n \)-ary functor and \( t_1, \ldots, t_n \) are terms then \( f(t_1, \ldots, t_n) \) is a term.

- The class of *atoms* (different from LP!):
  - a proposition is an atom,
  - if \( p \) is an \( n \)-ary pred. symbol and \( t_1, \ldots, t_n \) are terms then \( p(t_1, \ldots, t_n) \) is an atom,
  - true and false are atoms.

- The class of Well Formed Formulas (WFFs):
  - an atom is a WFF,
  - if \( F \) and \( G \) are WFFs then so are \( \neg F, (F \lor G), (F \land G), (F \rightarrow G) \) and \( (F \leftrightarrow G) \),
  - if \( F \) is a WFF and \( X \) is a variable then \( \exists X F \) and \( \forall X F \) are WFF.

- Literal: positive or negative (non-negated or negated) atom.
Examples

Examples of Terms

• Given:
  ◦ constants: a, b, c, 1, spot, john...
  ◦ functors: f/1, g/3, h/2, +/3...
  ◦ variables: X, L, Y...

• Correct: spot, f(john), f(X), +(1,2,3), +(X,Y,L), f(f(spot)), h(f(h(1,2)),L)

• Incorrect: spot(X), +(1,2), g, f(f(h))

Examples of Literals

• Given the elements above and:
  ◦ predicate symbols: dog/1, p/2, q/0, r/0, barks/1...

• Correct: q, r, dog(spot), p(X,f(john))...

• Incorrect: q(X), barks(f), dog(barks(X))
Examples of WFFs

- Given the elements above

- Correct: $q, q \rightarrow r, r \leftarrow q, \text{dog}(X) \leftarrow \text{barks}(X), \text{dog}(X), p(X, Y), \exists X (\text{dog}(X) \land \text{barks}(X) \land \neg q), \exists Y (\text{dog}(Y) \rightarrow \text{bark}(Y))$

- Incorrect: $q \lor, \exists p$
More about WFFs

- Allow us to represent knowledge and reason about it
  - Marcus was a man
  - Marcus was a pompeian
  - All pompeians were romans
  - Caesar was a ruler
  - All romans were loyal to Caesar or they hated him
  - Everyone is loyal to someone

We can now reason about this knowledge using standard deductive mechanisms.

But there is in principle no guarantee that we will prove a given theorem.
Towards Efficient Automated Deduction

- *Automated deduction is search.*
- Complexity of search: directly dependent on branching factor at nodes (exponentially!).
- It is vital to cut down the branching factor:
  - Canonical representation of nodes (allows identifying identical nodes).
  - As few inference rules as possible.
Towards Efficient Automated Deduction (Contd.)

Clausal Form

- The complete set of logical operators (←, ∧, ∨, ¬,...) is redundant.
- A minimal (canonical) form would be interesting.
- It would be interesting to separate the quantifiers from the rest of the formula so that they did not need to be considered.
- It would also be nice if the formula were flat (i.e. no parenthesis).
- Conjunctive normal form has these properties [Davis 1960].

Deduction Mechanism

- A good example:
  Resolution – only two inference rules (*Resolution rule* and *Replacement rule*).
Classical Clausal Form: Conjunctive Normal Form

- General formulas are converted to:
  - Set of *Clauses*.
  - Clauses are in a logical conjunction.
  - A clause is a disjunction of the form: \( \text{literal}_1 \lor \text{literal}_2 \lor \ldots \lor \text{literal}_n \)
  - The \( \text{literal}_i \) are negated or non-negated atoms.
  - All variables are implicitly universally quantified: i.e. if \( X_1, \ldots, X_k \) are the variables that appear in a clause it represents the formula: \( \forall X_1, \ldots, X_k \ \text{literal}_1 \lor \text{literal}_2 \lor \ldots \lor \text{literal}_n \)

- Any formula can be converted to clausal form automatically by:
  1. Converting to Prenex form.
  2. Converting to conjunctive normal form (conjunction of disjunctions).
  3. Converting to Skolem form (eliminating existential quantifiers).
  4. Eliminating universal quantifiers.
  5. Separating conjunctions into clauses.

- The *unsatisfiability* of a system is preserved.
Substitutions

- A substitution is a finite mapping from variables to terms, written as 
  \[ \theta = \{X_1/t_1, \ldots, X_n/t_n\} \] where
  - the variables \( X_1, \ldots, X_n \) are different,
  - for \( i = 1, \ldots, n \) \( X_i \not\equiv t_i \).

- A pair \( X_i/t_i \) is called a binding.

- \( \text{domain}(\theta) = \{X_1, \ldots, X_n\} \) and \( \text{range}(\theta) = \text{vars}(\{t_1, \ldots, t_n\}) \).

- If \( \text{range}(\theta) = \emptyset \) then \( \theta \) is called ground.

- If \( \theta \) is a bijective mapping from variables to variables then \( \theta \) is called a renaming.

- Examples:
  - \( \theta_1 = \{X/f(A), Y/X, Z/h(b, Y), W/a\} \)
  - \( \theta_2 = \{X/a, Y/a, Z/h(b, c), W/f(d)\} \) (ground)
  - \( \theta_3 = \{X/A, Y/B, Z/C, W/D\} \) (renaming)
Substitutions (Contd.)

- Substitutions operate on *expressions*, i.e. a term, a sequence of literals or a clause, denoted by $E$.
- The application of $\theta$ to $E$ (denoted $E\theta$) is obtained by *simultaneously* replacing each occurrence in $E$ of $X_i$ by $t_i$, $X_i/t_i \in \theta$.
- The resulting expression $E\theta$ is called an *instance* of $E$.
- If $\theta$ is a renaming then $E\theta$ is called a *variant* of $E$.
- Example:
  $\theta_1 = \{X/f(A), Y/X, Z/h(b,Y), W/a\}$
  $p(X, Y, X) \theta_1 = p(f(A), X, f(A))$
Composition of Substitutions

- Given $\theta = \{X_1/t_1, ..., X_n/t_n\}$ and $\eta = \{Y_1/s_1, ..., Y_m/s_m\}$ their composition $\theta \eta$ is defined by removing from the set

  \[ \{X_1/t_1\eta, ..., X_n/t_n\eta, Y_1/s_1, ..., Y_m/s_m\} \]

  those pairs $X_i/t_i\eta$ for which $X_i \equiv t_i\eta$, as well as those pairs $Y_i/s_i$ for which $Y_i \in \{X_1, ..., X_n\}$.

- Example: if $\theta = \{X/3, Y/f(X, 1)\}$ and $\eta = \{X/4\}$ then $\theta \eta = \{X/3, Y/f(4, 1)\}$.

- For all substitutions $\theta, \eta$ and $\gamma$ and an expression $E$

  i) $(E\theta)\eta \equiv E(\theta \eta)$

  ii) $(\theta \eta)\gamma = \theta(\eta \gamma)$.

- $\theta$ is more general than $\eta$ if for some $\gamma$ we have $\eta = \theta \gamma$.

- Example: $\theta = \{X/f(Y)\}$ more general than $\eta = \{X/f(h(G))\}$
Unifiers

- If $A\theta \equiv B\theta$, then
  - $\theta$ is called a unifier of $A$ and $B$
  - $A$ and $B$ are unifiable

- A unifier $\theta$ of $A$ and $B$ is called a most general unifier (mgu) if it is more general than any other unifier of $A$ and $B$.

- If two atoms are unifiable then they have a most general unifier.

- $\theta$ is idempotent if $\theta\theta = \theta$.

- A unifier $\theta$ of $A$ and $B$ is relevant if all variables appearing either in $\text{domain}(\theta)$ or in $\text{range}(\theta)$, also appear in $A$ or $B$.

- If two atoms are unifiable then they have an mgu which is idempotent and relevant.

- An mgu is unique up to renaming.
Unification Algorithm

- Non-deterministically choose from the set of equations an equation of a form below and perform the associated action.

1. \( f(s_1, ..., s_n) = f(t_1, ..., t_n) \) → replace by \( s_1 = t_1, ..., s_n = t_n \)
2. \( f(s_1, ..., s_n) = g(t_1, ..., t_m) \) where \( f \not\equiv g \) → halt with failure
3. \( X = X \) → delete the equation
4. \( t = X \) where \( t \) is not a variable → replace by the equation \( X = t \)
5. \( X = t \) where \( X \not\equiv t \) and \( X \) has another occurrence in the set of equations →
   5.1 if \( X \) appears in \( t \) then halt with failure
   5.2 otherwise apply \( \{X/t\} \) to every other equation

- Consider the set of equations \( \{f(x) = f(f(z)), g(a, y) = g(a, x)\} \):
  - (1) produces \( \{x = f(z), g(a, y) = g(a, x)\} \)
  - then (1) yields \( \{x = f(z), a = a, y = x\} \)
  - (3) produces \( \{x = f(z), y = x\} \)
  - now only (5) can be applied, giving \( \{x = f(z), y = f(z)\} \)
  - No step can be applied, the algorithm successfully terminates.
Unification Algorithm revisited

- Let $A$ and $B$ be two formulas:

1. $\theta = \epsilon$

2. while $A\theta \neq B\theta$:
   2.1 find leftmost symbol in $A\theta$ s.t. the corresponding symbol in $B\theta$ is different
   2.2 let $t_A$ and $t_B$ be the terms in $A\theta$ and $B\theta$ starting with those symbols
      (a) if neither $t_A$ nor $t_B$ are variables or one is a variable occurring in the other → halt with failure
      (b) otherwise, let $t_A$ be a variable → the new $\theta$ is the result of $\theta\{t_A/t_B\}$

3. end with $\theta$ being an m.g.u. of $A$ and $B$
Unification Algorithm revisited (Contd.)

- Example: \( A = p(X, X) \ B = p(f(A), f(B)) \)

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( A\theta )</th>
<th>( B\theta )</th>
<th>Element</th>
</tr>
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<tbody>
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<td>( p(f(A), f(B)) )</td>
<td>( {X/f(A)} )</td>
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<td>( p(f(A), f(B)) )</td>
<td>( {A/B} )</td>
</tr>
<tr>
<td>( {X/f(B), A/B} )</td>
<td>( p(f(B), f(B)) )</td>
<td>( p(f(B), f(B)) )</td>
<td></td>
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- Example: \( A = p(X, f(Y)) \ B = p(Z, X) \)

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Resolution with Variables

- It is a *formal system* with:
  - A first order language with the following formulas:
    - * Clauses: without repetition, and without an order among their literals.
    - * The empty clause $\Box$.
  - An empty set of axioms.
  - Two inference rules: *resolution* and *replacement*.
Resolution with Variables (Contd.)

• Resolution:

\[ r_1: A \lor F_1 \lor \cdots \lor F_n \]
\[ r_2: \neg B \lor G_1 \lor \cdots \lor G_m \]

\[
\frac{}{((F_1 \lor \cdots \lor F_n)\sigma \lor G_1 \lor \cdots \lor G_m)\theta}
\]

where

◊ A and B are unifiable with substitution \( \theta \)
◊ \( \sigma \) is a renaming s.t. \((A \lor F_1 \lor \cdots \lor F_n)\sigma\) and \(\neg B \lor G_1 \lor \cdots \lor G_m\) have no variables in common
◊ \( \theta \) is the m.g.u. of \( A\sigma \) and \( B \)

The resulting clause is called the *resolvent* of \( r_1 \) and \( r_2 \).

• Replacement: \( A \lor B \lor F_1 \lor \cdots \lor F_n \Rightarrow (A \lor F_1 \lor \cdots \lor F_n)\theta \) where

◊ A and B are unifiable atoms
◊ \( \theta \) is the m.g.u. of A and B
Basic Properties

- Resolution is *correct* – i.e. all conclusions obtained using it are valid.
- There is no guarantee of directly deriving a given theorem.
- However, resolution (under certain assumptions) is refutation complete: if we have a set of clauses \( K = [C_0, C_1, \ldots, C_n] \) and it is inconsistent then resolution will arrive at the empty clause \( \Box \) in a finite number of steps.
- Therefore, a valid theorem (or a question that has an answer) is guaranteed to be provable by refutation. To prove “p” given \( K_0 = [C_0, C_1, \ldots, C_n] \):
  1. Negate it (\( \neg p \)).
  2. Construct \( K = [\neg p, C_0, C_1, \ldots, C_n] \).
  3. Apply resolution steps repeatedly to \( K \).
- Furthermore, we can obtain answers by composing the substitutions along a path that leads to \( \Box \) (very important for realizing Green's dream!).
- It is important to use a good method in applying the resolution steps – i.e. in building the resolution tree (or proof tree).
- Again, the main issue is to reduce the branching factor.
Proof Tree

• Given a set of clauses \( K = \{ C_0, C_1, \cdots, C_n \} \) the proof tree of \( K \) is a tree s.t.:
  ◦ the root is \( C_0 \)
  ◦ the branch from the root starts with the nodes labeled with \( C_0, C_1, \cdots, C_n \)
  ◦ the descendent nodes of \( C_n \) are labeled by clauses obtained from the parent clauses using resolution
  ◦ a derivation in \( K \) is a branch of the proof tree of \( K \)

• The derivation \( C_0 C_1 \cdots C_n F_0 \cdots F_m \) is denoted as \( K, F_0 \cdots F_m \)
Proof Tree (Contd.)

- Example: part of the proof tree for K, with:

\[ K = \{ p, \neg p \vee q, \neg q \} \]

\[ \begin{align*}
  p & \equiv C_0 \\
  \neg p \vee q & \equiv C_1 \\
  \neg q & \equiv C_2 \\
\end{align*} \]

\[ R(C_0, C_1) \equiv q \]
\[ \neg p \equiv R(C_1, C_2) \]
\[ R(C_0, C_1) \equiv q \]
Characteristics of the Proof Tree

- It can be infinite:

\[ K = \left[ p(e), \neg p(X) \lor p(f(X)) \right] \]

\[
\begin{align*}
p(e) \equiv C_0 \\
\neg p(X) \lor p(f(X)) \equiv C_1 \\
p(f(e)) \quad \Theta = \{X/e\} \\
p(f(f(e))) \quad \Theta = \{X/f(e)\}
\end{align*}
\]

- Even if it is finite, it can be too large to be explored efficiently

- Aim: determine some criteria to limit the number of derivations and the way in which the tree is explored \( \Rightarrow \) strategy

- Any strategy based on this tree is correct: if \( \Box \) appears in a subtree of the proof tree of \( K \), then \( \Box \) can be derived from \( K \) and therefore \( K \) is unsatisfiable
General Strategies

- Depth-first with backtracking: First descendant to the left; if failure or ✗ then backtrack
General Strategies (Contd.)

- **Breadth first**: all sons of all sibling nodes from left to right

```
  1
 / \  \
2    3
|    |
4    5
|    |
8     Fail
|    |
9     Fail
```

1 2 3 Fail Fail Fail 6 7
• Iterative deepening
  ◦ Advance depth-first for a time.
  ◦ After a certain depth, switch to another branch as in breadth-first.

• Completeness issues / possible types of branches:
  ◦ Success (always finite)
  ◦ Finite failure
  ◦ Infinite failure (provably infinite branches)
  ◦ Non-provably infinite branches
Linear Strategies

- Those which only explore linear derivations

- A derivation $K, F_0 \cdots F_m$ is linear if
  
  - $F_0$ is obtained by resolution or replacement using $C_0$
  - $F_i, i > 0$ is obtained by resolution or replacement using $F_{i-1}$

- Examples:

  \[
  \begin{array}{c}
  p \equiv C0 \\
  \neg p \lor q \equiv C1 \\
  \neg q \equiv C2 \\
  q \equiv F0 \\
  \hline
  \end{array}
  \]

  \[
  \begin{array}{c}
  \neg p \lor q \equiv C0 \\
  p \equiv C1 \\
  \neg q \equiv C2 \\
  \neg p \quad q \\
  \hline
  \end{array}
  \]
Characteristics of these Strategies

1. If \( \square \) can be derived from \( K \) by using resolution with variables, it can also be derived by linear resolution.

2. Let \( K \) be \( K' \cup \{C_0\} \) where \( K' \) is a satisfiable set of clauses, i.e. \( \square \) cannot be derived from \( K' \) by using resolution with variables. If \( \square \) can be derived from \( K \) by using resolution with variables it can also be derived by linear resolution with root \( C_0 \).

- From (1), if the strategy is breadth first, it is complete.
- From (2), if we want to prove that \( B \) is derived form \( K' \) then we can apply linear resolution to \( K = K' \cup \{-B\} \).
- Depth first with backtracking is not complete:
Input Strategies

- Those which only explore input derivations
- A derivation $K, F_0 \cdots F_m$ is input if
  - $F_0$ is obtained by resolution or replacement using $C_0$
  - $F_i, i > 0$ is obtained by resolution or replacement using at least a clause in $K$

Example:

$K = [\neg p \vee \neg q, p \vee \neg r, r, q \vee \neg s, s \vee q]$

\[
\begin{align*}
  &\neg p \vee \neg q \quad \equiv C_0 \\
  &p \vee \neg r \quad \equiv C_1 \\
  &r \quad \equiv C_2 \\
  &q \vee \neg s \quad \equiv C_3 \\
  &s \vee q \quad \equiv C_4 \\
  &\neg q \vee \neg r \quad \equiv C_1 (\& C_0)
\end{align*}
\]

Input + Linear
Input Strategies

- In an input derivation, if $F_{i-1}$ does not appear in any derivation of a successor clause, it can be eliminated from the derivation without changing the result.
- If $F_{i-1}$ appears in the derivation of $F_j, j > 1$, $F_{i-1}$ can be allocated in position $j - 1$.
- As a result, we can limit ourselves to linear input derivations without losing any input derivable clause.
- Let $K$ be $K^\downarrow \cup \{C_0\}$ where $\Box$ is derived by using resolution with variables, $C_0$ is a negative Horn clause and all clauses in $K^\downarrow$ are positive Horn clauses. There is an input derivation with root $C_0$ finishing in $\Box$ and in which the replacement rule is not used (Hernschen 1974).
- A Horn clause is a clause in which at most one literal is positive:
  - it is positive if precisely one literal is positive
  - it is negative if all literals are negatives
- As a result, in those conditions, a breadth first input strategy is complete, and a depth first input strategy with backtracking is complete if the tree is finite.
Ordered Strategies

- We consider a new formal system in which:
  1. clauses are *ordered* sets
  2. ordered resolution of two clauses
     \[ A = p_1 \lor \cdots \lor p_n \text{ and } B = q_1 \lor \cdots \lor q_m \]
     where \( p_1 \) is a positive literal and \( q_1 \) is a negative literal is possible iff \( \neg p_1 \) and \( \sigma(q_1) \) are unifiable (\( \sigma \) is a renaming, s.t. \( p_1 \) and \( \sigma(q_1) \) have no variables in common)
  3. the resolvent of \( A \) and \( B \) is \( \theta(p_2 \lor \cdots \lor p_n \lor \sigma(q_2 \lor \cdots \lor q_m)) \) where \( \theta \) is an m.g.u of \( \neg p_1 \) and \( \sigma(q_1) \)

- Let \( K = K' \cup \{C_0\} \) be a set of clauses s.t. \( \square \) is derived by using resolution with variables, \( C_0 \) is a negative Horn clause and all clauses in \( K' \) are positive Horn clauses with the positive literal in the first place. There is a sorted input derivation with root \( C_0 \) arriving at \( \square \).

- In this context a sorted linear input with:
  - breadth first: is complete
  - depth first with backtracking: is complete if the tree is finite