Computational Logic
Automated Deduction Fundamentals
Elements of First-Order Predicate Logic

First Order Language:

- An *alphabet* consists of the following classes of symbols:
  1. *variables* denoted by $X, Y, Z, Boo, ...$, (infinite)
  2. *constants* denoted by $1, a, boo, john, ...$
  3. *functors* denoted by $f, g, +, −, ..$
  4. *predicate symbols* denoted by $p, q, dog, ...$
  5. *connectives*, which are: $\neg$ (negation), $\lor$ (disjunction), $\land$ (conjunction), $\rightarrow$ (implication) and $\leftrightarrow$ (equivalence),
  6. *quantifiers*, which are: $\exists$ (there exists) and $\forall$ (for all),
  7. *parentheses*, which are: ( and ) and the comma, that is: “,”.

- Each functor and predicate symbol has a fixed *arity*, they are often represented in *Functor/Arity* form, e.g. $f/3$.
- A constant can be seen as a functor of arity 0.
- Propositions are represented by a predicate symbol of arity 0.
Important: Notation Convention Used

(A bit different from standard notational conventions in logic, but good for compatibility with LP systems)

- Variables: start with a capital letter or a “_” (X, Y, _a, _1)
- Atoms, functors, predicate symbols: start with a lower case letter or are enclosed in ’ ’ (f, g, a, 1, x, y, z, ’X’, ’_1’)
Terms and Atoms

We define by induction two classes of strings of symbols over a given alphabet.

- **The class of terms:**
  
  ◦ a variable is a term,
  ◦ a constant is a term,
  ◦ if $f$ is an $n$-ary functor and $t_1, \ldots, t_n$ are terms then $f(t_1, \ldots, t_n)$ is a term.

- **The class of atoms (different from LP!):**
  
  ◦ a proposition is an atom,
  ◦ if $p$ is an $n$-ary pred. symbol and $t_1, \ldots, t_n$ are terms then $p(t_1, \ldots, t_n)$ is an atom,
  ◦ true and false are atoms.

- **The class of Well Formed Formulas (WFFs):**
  
  ◦ an atom is a WFF,
  ◦ if $F$ and $G$ are WFFs then so are $\neg F$, $(F \lor G)$, $(F \land G)$, $(F \rightarrow G)$ and $(F \leftrightarrow G)$,
  ◦ if $F$ is a WFF and $X$ is a variable then $\exists X \, F$ and $\forall X \, F$ are WFF.

- **Literal:** positive or negative (non-negated or negated) atom.
Examples

Examples of Terms

- Given:
  - constants: \(a, b, c, 1, \text{spot, john}...\)
  - functors: \(f/1, g/3, h/2, +/3...\)
  - variables: \(X, L, Y...\)
- Correct: \(\text{spot}, f(\text{john}), f(X), +(1,2,3), +(X,Y,L), f(f(\text{spot})), h(f(h(1,2)),L)\)
- Incorrect: \(\text{spot}(X), +(1,2), g, f(f(h))\)

Examples of Literals

- Given the elements above and:
  - predicate symbols: \(\text{dog/1, p/2, q/0, r/0, barks/1}...\)
- Correct: \(q, r, \text{dog(\text{spot})}, p(\text{X},f(\text{john})))...\)
- Incorrect: \(q(\text{X}), \text{barks(f), dog(barks(X))}\)
Examples of WFFs

- Given the elements above

- Correct: $q, q \rightarrow r, r \leftarrow q, \text{dog}(X) \leftarrow \text{barks}(X), \text{dog}(X), p(X,Y), \exists X (\text{dog}(X) \land \text{barks}(X) \land \neg q), \exists Y (\text{dog}(Y) \rightarrow \text{bark}(Y))$

- Incorrect: $q \lor, \exists p$

Examples (Contd.)
More about WFFs

- Allow us to represent knowledge and reason about it
  - Marcus was a man: \textit{man(marcus)}
  - Marcus was a Pompeian: \textit{pompeian(marcus)}
  - All Pompeians were Romans: \( \forall X \ \text{pompeian}(X) \rightarrow \text{roman}(X) \)
  - Caesar was a ruler: \textit{ruler(caesar)}
  - All Romans were loyal to Caesar or they hated him:
    \( \forall X \ \text{roman}(X) \rightarrow \text{loyalto}(X, \text{caesar}) \lor \text{hate}(X, \text{caesar}) \)
  - Everyone is loyal to someone:
    \( \forall X \ \exists Y \ \text{loyalto}(X, Y) \)

- We can now reason about this knowledge using standard deductive mechanisms.
- But there is in principle no guarantee that we will prove a given theorem.
Towards Efficient Automated Deduction

- *Automated deduction is search.*
- Complexity of search: directly dependent on branching factor at nodes (exponentially!).
- It is vital to cut down the branching factor:
  - Canonical representation of nodes (allows identifying identical nodes).
  - As few inference rules as possible.
Towards Efficient Automated Deduction (Contd.)

Clausal Form

- The complete set of logical operators ($\leftarrow, \land, \lor, \neg,...$) is redundant.
- A minimal (canonical) form would be interesting.
- It would be interesting to separate the quantifiers from the rest of the formula so that they did not need to be considered.
- It would also be nice if the formula were flat (i.e. no parenthesis).
- Conjunctive normal form has these properties [Davis 1960].

Deduction Mechanism

- A good example:
  Resolution – only two inference rules (Resolution rule and Replacement rule).
Classical Clausal Form: Conjunctive Normal Form

- General formulas are converted to:
  - Set of *Clauses*.
  - Clauses are in a logical conjunction.
  - A clause is a disjunction of the form: $\text{literal}_1 \lor \text{literal}_2 \lor \ldots \lor \text{literal}_n$
  - The $\text{literal}_i$ are negated or non-negated atoms.
  - All variables are implicitly universally quantified: i.e. if $X_1, \ldots, X_k$ are the variables that appear in a clause it represents the formula:
  $$\forall X_1, \ldots, X_k \text{ literal}_1 \lor \text{literal}_2 \lor \ldots \lor \text{literal}_n$$

- Any formula can be converted to clausal form automatically by:
  1. Converting to Prenex form.
  2. Converting to conjunctive normal form (conjunction of disjunctions).
  3. Converting to Skolem form (eliminating existential quantifiers).
  4. Eliminating universal quantifiers.
  5. Separating conjunctions into clauses.

- The *unsatisfiability* of a system is preserved.
Substitutions

- A substitution is a finite mapping from variables to terms, written as 
  \( \theta = \{ X_1/t_1, ..., X_n/t_n \} \) where
  
  ◦ the variables \( X_1, ..., X_n \) are different,
  ◦ for \( i = 1, ..., n \) \( X_i \not\equiv t_i \).

- A pair \( X_i/t_i \) is called a binding.

- \( \text{domain}(\theta) = \{ X_1, .., X_n \} \) and \( \text{range}(\theta) = \text{vars}(\{t_1, ..., t_n\}) \).

- If \( \text{range}(\theta) = \emptyset \) then \( \theta \) is called ground.

- If \( \theta \) is a bijective mapping from variables to variables then \( \theta \) is called a renaming.

- Examples:
  
  ◦ \( \theta_1 = \{ X/f(A), Y/X, Z/h(b, Y), W/a \} \)
  ◦ \( \theta_2 = \{ X/a, Y/a, Z/h(b, c), W/f(d) \} \) (ground)
  ◦ \( \theta_3 = \{ X/A, Y/B, Z/C, W/D \} \) (renaming)
Substitutions operate on expressions, i.e. a term, a sequence of literals or a clause, denoted by $E$.

The application of $\theta$ to $E$ (denoted $E\theta$) is obtained by simultaneously replacing each occurrence in $E$ of $X_i$ by $t_i$, $X_i/t_i \in \theta$.

The resulting expression $E\theta$ is called an instance of $E$.

If $\theta$ is a renaming then $E\theta$ is called a variant of $E$.

Example:

$$\theta_1 = \{X/f(A), Y/X, Z/h(b, Y), W/a\}$$

$$p(X, Y, X) \theta_1 = p(f(A), X, f(A))$$
Composition of Substitutions

• Given \( \theta = \{ X_1/t_1, ..., X_n/t_n \} \) and \( \eta = \{ Y_1/s_1, ..., Y_m/s_m \} \) their composition \( \theta \eta \) is defined by removing from the set

\[
\{ X_1/t_1 \eta, ..., X_n/t_n \eta, Y_1/s_1, ..., Y_m/s_m \}
\]

those pairs \( X_i/t_i \eta \) for which \( X_i \equiv t_i \eta \), as well as those pairs \( Y_i/s_i \) for which \( Y_i \in \{ X_1, ..., X_n \} \).

• Example: if \( \theta = \{ X/3, Y/f(X, 1) \} \) and \( \eta = \{ X/4 \} \) then \( \theta \eta = \{ X/3, Y/f(4, 1) \} \).

• For all substitutions \( \theta, \eta \) and \( \gamma \) and an expression \( E \)
  
  i) \( (E \theta) \eta \equiv E(\theta \eta) \)
  
  ii) \( (\theta \eta) \gamma = \theta(\eta \gamma) \).

• \( \theta \) is more general than \( \eta \) if for some \( \gamma \) we have \( \eta = \theta \gamma \).

• Example: \( \theta = \{ X/f(Y) \} \) more general than \( \eta = \{ X/f(h(G)) \} \)
Unifiers

• If $A\theta \equiv B\theta$, then
  ◦ $\theta$ is called a unifier of $A$ and $B$
  ◦ $A$ and $B$ are unifiable

• A unifier $\theta$ of $A$ and $B$ is called a most general unifier (mgu) if it is more general than any other unifier of $A$ and $B$.

• If two atoms are unifiable then they have a most general unifier.

• $\theta$ is idempotent if $\theta\theta = \theta$.

• A unifier $\theta$ of $A$ and $B$ is relevant if all variables appearing either in $\text{domain}(\theta)$ or in $\text{range}(\theta)$, also appear in $A$ or $B$.

• If two atoms are unifiable then they have an mgu which is idempotent and relevant.

• An mgu is unique up to renaming.
Unification Algorithm

• Non-deterministically choose from the set of equations an equation of a form below and perform the associated action.

1. \( f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n) \) → replace by \( s_1 = t_1, \ldots, s_n = t_n \)

2. \( f(s_1, \ldots, s_n) = g(t_1, \ldots, t_m) \) where \( f \not\equiv g \) → halt with failure

3. \( X = X \) → delete the equation

4. \( t = X \) where \( t \) is not a variable → replace by the equation \( X = t \)

5. \( X = t \) where \( X \not\equiv t \) and \( X \) has another occurrence in the set of equations →
   5.1 if \( X \) appears in \( t \) then halt with failure
   5.2 otherwise apply \( \{X/t\} \) to every other equation

• Consider the set of equations \( \{f(x) = f(f(z)), g(a, y) = g(a, x)\} \):
   ◊ (1) produces \( \{x = f(z), g(a, y) = g(a, x)\} \)
   ◊ then (1) yields \( \{x = f(z), a = a, y = x\} \)
   ◊ (3) produces \( \{x = f(z), y = x\} \)
   ◊ now only (5) can be applied, giving \( \{x = f(z), y = f(z)\} \)
   ◊ No step can be applied, the algorithm successfully terminates.
Unification Algorithm revisited

- Let $A$ and $B$ be two formulas:
  1. $\theta = \epsilon$
  2. while $A\theta \neq B\theta$:
     2.1 find leftmost symbol in $A\theta$ s.t. the corresponding symbol in $B\theta$ is different
     2.2 let $t_A$ and $t_B$ be the terms in $A\theta$ and $B\theta$ starting with those symbols
        (a) if neither $t_A$ nor $t_B$ are variables or one is a variable occurring in the other $\rightarrow$ halt with failure
        (b) otherwise, let $t_A$ be a variable $\rightarrow$ the new $\theta$ is the result of $\theta \{ t_A/t_B \}$
  3. end with $\theta$ being an m.g.u. of $A$ and $B$
Unification Algorithm revisited (Contd.)

- **Example:** \( A = p(X, X) \) \( B = p(f(A), f(B)) \)

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( A\theta )</th>
<th>( B\theta )</th>
<th>Element</th>
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<td>( p(f(A), f(B)) )</td>
<td>( {X/f(A)} )</td>
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<td>( p(f(A), f(B)) )</td>
<td>( {A/B} )</td>
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<tr>
<td>( {X/f(B), A/B} )</td>
<td>( p(f(B), f(B)) )</td>
<td>( p(f(B), f(B)) )</td>
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- **Example:** \( A = p(X, f(Y)) \) \( B = p(Z, X) \)

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Resolution with Variables

- It is a *formal system* with:
  - A first order language with the following formulas:
    - Clauses: without repetition, and without an order among their literals.
    - The empty clause $\square$.
  - An empty set of axioms.
  - Two inference rules: *resolution* and *replacement*. 
Resolution with Variables (Contd.)

- Resolution:

\[
\begin{align*}
  r_1 &\colon A \lor F_1 \lor \cdots \lor F_n \\
  r_2 &\colon \neg B \lor G_1 \lor \cdots \lor G_m \\
\end{align*}
\]

\[
\frac{((F_1 \lor \cdots \lor F_n)\sigma \lor G_1 \lor \cdots \lor G_m)\theta}{((F_1 \lor \cdots \lor F_n) \lor G_1 \lor \cdots \lor G_m)\theta}
\]

where

- \( A \) and \( B \) are unifiable with substitution \( \theta \)
- \( \sigma \) is a renaming s.t. \((A \lor F_1 \lor \cdots \lor F_n)\sigma \) and \( \neg B \lor G_1 \lor \cdots \lor G_m \) have no variables in common
- \( \theta \) is the m.g.u. of \( A\sigma \) and \( B \)

The resulting clause is called the resolvent of \( r_1 \) and \( r_2 \).

- Replacement: \( A \lor B \lor F_1 \lor \cdots \lor F_n \Rightarrow (A \lor F_1 \lor \cdots \lor F_n)\theta \) where

- \( A \) and \( B \) are unifiable atoms
- \( \theta \) is the m.g.u. of \( A \) and \( B \)
Basic Properties

- Resolution is *correct* – i.e. all conclusions obtained using it are valid.
- There is no guarantee of directly deriving a given theorem.
- However, resolution (under certain assumptions) is refutation complete: if we have a set of clauses $K = [C_0, C_1, \ldots, C_n]$ and it is inconsistent then resolution will arrive at the empty clause $\square$ in a finite number of steps.
- Therefore, a valid theorem (or a question that has an answer) is guaranteed to be provable by refutation. To prove “$p$” given $K_0 = [C_0, C_1, \ldots, C_n]$:
  1. Negate it ($\neg p$).
  2. Construct $K = [\neg p, C_0, C_1, \ldots, C_n]$.
  3. Apply resolution steps repeatedly to $K$.
- Furthermore, we can obtain answers by composing the substitutions along a path that leads to $\square$ (very important for realizing Greene’s dream!).
- It is important to use a good method in applying the resolution steps – i.e. in building the resolution tree (or proof tree).
- Again, the main issue is to reduce the branching factor.
Proof Tree

• Given a set of clauses $K = \{C_0, C_1, \cdots, C_n\}$ the proof tree of $K$ is a tree s.t.:
  ◦ the root is $C_0$
  ◦ the branch from the root starts with the nodes labeled with $C_0, C_1, \cdots, C_n$
  ◦ the descendent nodes of $C_n$ are labeled by clauses obtained from the parent clauses using resolution
  ◦ a derivation in $K$ is a branch of the proof tree of $K$

• The derivation $C_0C_1\cdots C_nF_0\cdots F_m$ is denoted as $K, F_0\cdots F_m$
Proof Tree (Contd.)

- Example: part of the proof tree for $K$, with:

$$K = [p, \neg p \vee q, \neg q]$$

- $p \equiv C_0$
- $\neg p \vee q \equiv C_1$
- $\neg q \equiv C_2$

- $R(C_0, C_1) \equiv q$
- $\neg p \equiv R(C_1, C_2)$

Diagram representation of the proof tree:

- $p \equiv C_0$
- $\neg p \equiv R(C_1, C_2)$
- $\neg q \equiv C_2$
- $R(C_0, C_1) \equiv q$

[Diagram showing the proof tree with nodes and edges labeled accordingly]
Characteristics of the Proof Tree

• It can be infinite:

\[ K = [ p(e), \neg p(X) \lor p(f(X)) ] \]

\[ \begin{align*}
     p(e) & \quad \equiv \quad C0 \\
     \neg p(X) \lor p(f(X)) & \quad \equiv \quad C1 \\
     p(f(e)) & \\
     p(f(f(e))) & \Theta = \{ X/e \} \\
     \ldots & \Theta = \{ X/f(e) \}
\end{align*} \]

• Even if it is finite, it can be too large to be explored efficiently

• Aim: determine some criteria to limit the number of derivations and the way in which the tree is explored \( \Rightarrow \) strategy

• Any strategy based on this tree is correct: if \( \Box \) appears in a subtree of the proof tree of \( K \), then \( \Box \) can be derived from \( K \) and therefore \( K \) is unsatisfiable
General Strategies

- **Depth-first with backtracking:** First descendant to the left; if failure or □ then backtrack
• **Breadth first**: all sons of all sibling nodes from left to right
• **Iterative deepening**
  ◦ Advance depth-first for a time.
  ◦ After a certain depth, switch to another branch as in breadth-first.

• **Completeness issues / possible types of branches:**
  ◦ Success (always finite)
  ◦ Finite failure
  ◦ Infinite failure (provably infinite branches)
  ◦ Non-provably infinite branches
Linear Strategies

- Those which only explore linear derivations
- A derivation $K, F_0 \cdots F_m$ is linear if
  - $F_0$ is obtained by resolution or replacement using $C_0$
  - $F_i, i < 0$ is obtained by resolution or replacement using $F_{i-1}$
- Examples:

```
\begin{align*}
  p & \equiv C0 \\
  \neg p \lor q & \equiv C1 \\
  \neg q & \equiv C2 \\
  q & \equiv F0 \\
\end{align*}
```

```
\begin{align*}
  \neg p \lor q & \equiv C0 \\
  p & \equiv C1 \\
  \neg q & \equiv C2 \\
  \neg p & \quad q \\
\end{align*}
```
1 If □ can be derived from $K$ by using resolution with variables, it can also be derived by linear resolution.

2 Let $K$ be $K' \cup \{C_0\}$ where $K'$ is a satisfiable set of clauses, i.e. □ cannot be derived from $K'$ by using resolution with variables. If □ can be derived from $K$ by using resolution with variables it can also be derived by linear resolution with root $C_0$.

- From (1), if the strategy is breadth first, it is complete.
- From (2), if we want to prove that $B$ is derived form $K'$ then we can apply linear resolution to $K = K' \cup \{\neg B\}$.
- Depth first with backtracking is not complete:

\[
K = \{ p(e), \neg p(X) \lor p(f(X)), \neg p(X) \} \\
p(e) \equiv C_0 \\
\neg p(X) \lor p(f(X)) \equiv C_1 \\
\neg p(X) \equiv C_2 \\
F_0 \equiv p(f(e)) \\
F_1 \equiv p(f(f(e))) \\
\square
\]
Input Strategies

- Those which only explore input derivations
- A derivation $K, F_0 \cdots F_m$ is input if
  - $F_0$ is obtained by resolution or replacement using $C_0$
  - $F_i, i < 0$ is obtained by resolution or replacement using at least a clause in $K$

Example:

$K = [\neg p \lor q, p \lor \neg r, r, q \lor \neg s, s \lor q]$

\[
\begin{align*}
\neg p \lor q & \quad \equiv \quad C_0 \\
p \lor \neg r & \quad \equiv \quad C_1 \\
\neg r & \quad \equiv \quad C_2 \\
q \lor \neg s & \quad \equiv \quad C_3 \\
s \lor q & \quad \equiv \quad C_4
\end{align*}
\]

- $\neg q \lor \neg r \quad \equiv \quad C_1 \quad (\& C_0)$

Input + Linear
Input Strategies

• In an input derivation, if $F_{i-1}$ does not appear in any derivation of a successor clause, it can be eliminated from the derivation without changing the result.

• If $F_{i-1}$ appears in the derivation of $F_j$, $j > 1$, $F_{i-1}$ can be allocated in position $j - 1$.

• As a result, we can limit ourselves to linear input derivations without losing any input derivable clause.

• Let $K$ be $K' \cup \{ C_0 \}$ where $\square$ is derived by using resolution with variables, $C_0$ is a negative Horn clause and all clauses in $K'$ are positive Horn clauses. There is an input derivation with root $C_0$ finishing in $\square$ and in which the replacement rule is not used (Hernschen 1974).

• A Horn clause is a clause in which at most one literal is positive:
  ◦ it is positive if precisely one literal is positive
  ◦ it is negative if all literals are negatives

• As a result, in those conditions, a breadth first input strategy is complete, and a depth first input strategy with backtracking is complete if the tree is finite.
Ordered Strategies

• We consider a new formal system in which:
  1. clauses are ordered sets
  2. ordered resolution of two clauses
     \[ A = p_1 \lor \cdots \lor p_n \text{ and } B = q_1 \lor \cdots \lor q_m \]
     where \( p_1 \) is a positive literal and \( q_1 \) is a negative literal is possible iff \( \neg p_1 \) and \( \sigma(q_1) \) are unifiable (\( \sigma \) is a renaming, s.t. \( p_1 \) and \( \sigma(q_1) \) have no variables in common)
  3. the resolvent of \( A \) and \( B \) is \( \theta(p_2 \lor \cdots \lor p_n \lor \sigma(q_2 \lor \cdots \lor q_m)) \) where \( \theta \) is an m.g.u of \( \neg p_1 \) and \( \sigma(q_1) \)

• Let \( K = K' \cup \{C_0\} \) be a set of clauses s.t. \( \Box \) is derived by using resolution with variables, \( C_0 \) is a negative Horn clause and all clauses in \( K' \) are positive Horn clauses with the positive literal in the first place. There is a sorted input derivation with root \( C_0 \) arriving at \( \Box \).

• In this context a sorted linear input with:
  ◦ breadth first: is complete
  ◦ depth first with backtracking: is complete if the tree is finite