Computational Logic

A “Hands-on” Introduction to Pure Logic Programming

Syntax: Terms (Variables, Constants, and Structures)

(using Prolog notation conventions)

- **Variables:** start with uppercase character (or "_"), may include "_" and digits:

  Examples: `X, Im4u, A_little_garden, _, _x, _22`

- **Constants:** lowercase first character, may include "_" and digits. Also, numbers and some special characters. Quoted, any character:

  Examples: `a, dog, a_big_cat, 23, 'Hungry man', []`

- **Structures:** a **functor** (the structure name, is like a constant name) followed by a fixed number of arguments between parentheses:

  Example: `date(monday, Month, 1994)`

  Arguments can in turn be variables, constants and structures.

  ◇ **Arity:** is the number of arguments of a structure. Functors are represented as *name/arity*. A constant can be seen as a structure with arity zero.

Variables, constants, and structures as a whole are called **terms** (they are the terms of a "first–order language"): the *data structures* of a logic program.

Syntax: Terms

(using Prolog notation conventions)

- *Examples of terms*:

Term	*Type*	*Main functor:*
`dad`	constant	`dad/0`
`time(min, sec)`	structure	`time/2`
`pair(Calvin, tiger(Hobbes))`	structure	`pair/2`
`Tee(Alf, rob)`	illegal	—
`A_good_time`	variable	—

- *Functors* can be defined as *prefix*, *postfix*, or *infix* $\boxed{\textit{operators}}$ (just syntax!):

`a + b`	is the term	`'+'(a,b)`	if `+/2` declared infix
`- b`	is the term	`'-'(b)`	if `-/1` declared prefix
`a < b`	is the term	`'<'(a,b)`	if `</2` declared infix
`john father mary`	is the term	`father(john,mary)`	if `father/2` declared infix

We assume that some such operator definitions are always preloaded.

Syntax: Rules and Facts (Clauses)

- **Rule:** an expression of the form:

$$\boxed{\begin{array}{ll} p_0(t_1, t_2, \ldots, t_{n_0}) \leftarrow & p_1(t_1^1, t_2^1, \ldots, t_{n_1}^1), \\ & \ldots \\ & p_m(t_1^m, t_2^m, \ldots, t_{n_m}^m). \end{array}}$$

  - ◇ $p_0(...)$ to $p_m(...)$ are *syntactically* like *terms*.
  - ◇ $p_0(...)$ is called the **head** of the rule.
  - ◇ The p_i to the right of the arrow are called *literals* and form the **body** of the rule. They are also called **procedure calls**.
  - ◇ Usually, `:-` is called the **neck** of the rule.

- **Fact:** an expression of the form $\boxed{p(t_1, t_2, \ldots, t_n).}$ (i.e., a rule with empty body).

  Example:

```
meal(soup, beef, coffee).                  % ← A fact.
```

```
meal(First, Second, Third) :-              % ← A rule.
     appetizer(First),                     %
     main_dish(Second),                    %
     dessert(Third).                       %
```

- Rules and facts are both called **clauses**.

Syntax: Predicates, Programs, and Queries

- **Predicate** (or *procedure definition*): a set of clauses whose heads have the same name and arity (called the **predicate name**).

  Examples:

```
pet(spot).                              animal(spot).
pet(X) :- animal(X), barks(X).          animal(barry).
pet(X) :- animal(X), meows(X).          animal(hobbes).
```

  Predicate `pet/1` has three clauses. Of those, one is a fact and two are rules.
  Predicate `animal/1` has three clauses, all facts.

- **Logic Program:** a *set* of predicates.

- **Query:** an expression of the form: $\boxed{\leftarrow p_1(t_1^1, \ldots, t_{n_1}^1), \ldots, p_n(t_1^n, \ldots, t_{n_m}^n).}$
  (i.e., a clause without a head).
  A query represents a *question to the program*.
  Example: `:- pet(X).` In most systems written as: `?- pet(X).`

“Declarative” Meaning of Facts and Rules

The declarative meaning is the corresponding one in first order logic, according to certain conventions:

- **Facts**: state things that are true.
  (Note that a fact “`p.`” can be seen as the rule “ `p :- true.` ”)

  Example: the fact `animal(spot).`
  can be read as “spot is an animal”.

- **Rules:**
  - ◇ Commas in rule bodies represent conjunction, i.e.,
    $p \leftarrow p_1, \cdots, p_m.$ represents $p \leftarrow p_1 \wedge \cdots \wedge p_m.$
  - ◇ “$\leftarrow$” represents as usual logical implication.

  Thus, a rule $p \leftarrow p_1, \cdots, p_m.$ means “if p_1 and ... and p_m are true, then p is true”

  Example: the rule `pet(X):- animal(X), barks(X).`
  can be read as “X is a pet if it is an animal and it barks”.

"Declarative" Meaning of Predicates and Queries

- **Predicates**: clauses in the same predicate

  $p \leftarrow p_1, \ldots, p_n$

  $p \leftarrow q_1, \ldots, q_m$

  ...

  provide different *alternatives* (for p).

  Example: the rules

```
pet(X) :- animal(X), barks(X).
pet(X) :- animal(X), meows(X).
```

  express two ways for X to be a pet.

- **Note** (variable *scope*): the X vars. in the two clauses above are different, despite the same name. Vars. are *local to clauses* (and are *renamed* any time a clause is used –as with vars. local to a procedure in conventional languages).

- A **query** represents a *question to the program*.

  Examples:

```
?- pet(spot).
```

  asks whether spot is a pet.

```
?- pet(X).
```

  asks: "Is there an X which is a pet?"

“Execution” and Semantics

- Example of a **logic program**:

```
pet(X) :-  animal(X), barks(X).
pet(X) :-  animal(X), meows(X).
animal(spot).          barks(spot).
animal(barry).         meows(barry).
animal(hobbes).        roars(hobbes).
```

- **Execution:** given a program and a query, *executing* the logic program is *attempting to find an answer to the query*.

  Example: given the program above and the query `:- pet(X).`
  the system will try to find a “substitution” for `X` which makes `pet(X)` true.

  ◇ The **declarative semantics** specifies *what* should be computed (all possible answers).

  ⇒ Intuitively, we have two possible answers: `X = spot` and `X = barry`.

  ◇ The **operational semantics** specifies *how* answers are computed (which allows us to determine *how many steps* it will take).

Running Programs in a Logic Programming System

- File `pets.pl` contains (explained later):

```
:- module(_,_,['bf/bfall']).
```

  + the pet example code as in previous slides.

- Interaction with the system query evaluator (the “top level”):

```
?- Ciao 1.XX ...
?- use_module(pets).
yes
?- pet(spot).
yes
?- pet(X).
X = spot ? ;
X = barry ? ;
no
?-
```

See the part on Developing Programs with a Logic Programming System
for more details on the particular system used in the course (Ciao).

Simple (Top-Down) Operational Meaning of Programs

- A logic program is operationally a set of *procedure definitions* (the predicates).
- A query $\leftarrow \texttt{p}$ is an initial *procedure call*.
- A procedure definition with one *clause* $\texttt{p} \leftarrow \texttt{p}_1\texttt{,...,}\texttt{p}_m\texttt{.}$ means:
  "to execute a call to $\texttt{p}$ you have to *call* $\texttt{p}_1$ $\texttt{and}$ $\texttt{...and}$ $\texttt{p}_m$"
  - ◇ In principle, the order in which $\texttt{p}_1\texttt{, ..., p}_n$ are called does not matter, but, in practical systems it is fixed.
- If several clauses (definitions) means:

$$\texttt{p} \leftarrow \texttt{p}_1\texttt{, ..., p}_n$$
$$\texttt{p} \leftarrow \texttt{q}_1\texttt{, ..., q}_m$$
$$\ldots$$

  "to execute a call to $\texttt{p}$, call $\texttt{p}_1 \wedge \texttt{...} \wedge \texttt{p}_n$, or, alternatively, $\texttt{q}_1 \wedge \texttt{...} \wedge \texttt{q}_n$, or ..."
  - ◇ Unique to logic programming –it is like having several alternative procedure definitions.
  - ◇ Means that several possible paths may exist to a solution and they *should be explored*.
  - ◇ System usually stops when the first solution found, user can ask for more.
  - ◇ Again, in principle, the order in which these paths are explored does not matter (*if certain conditions are met*), but, for a given system, this is typically also fixed.

In the following we define a more precise operational semantics.

Unification: uses

- **Unification** is the mechanism used in *procedure calls* to:
  - ◇ Pass parameters.
  - ◇ “Return” values.
- It is also used to:
  - ◇ Access parts of structures.
  - ◇ Give values to variables.
- Unification is a procedure to solve equations on data structures.
  - ◇ As usual, it returns a minimal solution to the equation (or the equation system).
  - ◇ As many equation solving procedures it is based on isolating variables and then *instantiating* them with their values.

Unification

- **Unifying two terms (or literals) A and B:** is asking if they can be made syntactically identical by giving (minimal) values to their variables.
  - ◇ I.e., find a **variable substitution** θ such that $\boxed{A\theta = B\theta}$ (or, if impossible, *fail*).
  - ◇ Only variables can be given values!
  - ◇ Two structures can be made identical only by making their arguments identical.

E.g.:

A	B	θ	$A\theta$	$B\theta$
dog	dog	$\emptyset$	dog	dog
`X`	`a`	{`X` = `a`}	`a`	`a`
`X`	`Y`	{`X` = `Y`}	`Y`	`Y`
`f(X, g(t))`	`f(m(h), g(M))`	{`X=m(h), M=t`}	`f(m(h), g(t))`	`f(m(h), g(t))`
`f(X, g(t))`	`f(m(h), t(M))`	Impossible (1)		
`f(X, X)`	`f(Y, l(Y))`	Impossible (2)		

- (1) Structures with different name and/or arity cannot be unified.
- (2) A variable cannot be given as value a term which contains that variable, because it would create an infinite term. This is known as the **occurs check**. (See, however, *cyclic terms* later.)

Unification

- Often several solutions exist, e.g.:

A	B	θ_1	$A\theta_1$ and $B\theta_1$
`f(X, g(T))`	`f(m(H), g(M))`	{ `X`=`m(a)`, `H`=`a`, `M`=`b`, `T`=`b` }	`f(m(a), g(b))`
”	”	{ `X`=`m(H)`, `M`=`f(A)`, `T`=`f(A)` }	`f(m(H), g(f(A)))`

These are correct, but a simpler (“more general”) solution exists:

A	B	θ_1	$A\theta_1$ and $B\theta_1$
`f(X, g(T))`	`f(m(H), g(M))`	{ `X`=`m(H)`, `T`=`M` }	`f(m(H), g(M))`

- Always a unique (modulo variable renaming) *most general* solution exists (unless unification fails).
- This is the one that we are interested in.
- The *unification algorithm* finds this solution.

Unification Algorithm

- Let A and B be two terms:

  1 $\theta = \emptyset$, $E = \{A = B\}$

  2 while not $E = \emptyset$:

  2.1 delete an equation $T = S$ from E

  2.2 case T or S (or both) are (distinct) variables. Assuming T variable:

  * (occur check) if T occurs in the term $S \rightarrow$ halt with failure
  * substitute variable T by term S in all terms in θ
  * substitute variable T by term S in all terms in E
  * add $T = S$ to θ

  2.3 case T and S are non-variable terms:

  * if their names or arities are different $\rightarrow$ halt with failure
  * obtain the arguments $\{T_1, \ldots, T_n\}$ of T and $\{S_1, \ldots, S_n\}$ of S
  * add $\{T_1 = S_1, \ldots, T_n = S_n\}$ to E

  3 halt with θ being the m.g.u of A and B

Unification Algorithm Examples (I)

- Unify: A = `p(X,X)` and B = `p(f(Z),f(W))`

θ	E	T	S
{}	{ p(X,X)=p(f(Z),f(W)) }	p(X,X)	p(f(Z),f(W))
{}	{ X=f(Z), X=f(W) }	X	f(Z)
{ X=f(Z) }	{ f(Z)=f(W) }	f(Z)	f(W)
{ X=f(Z) }	{ Z=W }	Z	W
{ X=f(W), Z=W }	{}		

- Unify: A = `p(X,f(Y))` and B = `p(Z,X)`

θ	E	T	S
{}	{ p(X,f(Y))=p(Z,X) }	p(X,f(Y))	p(Z,X)
{}	{ X=Z, f(Y)=X }	X	Z
{ X=Z }	{ f(Y)=Z }	f(Y)	Z
{ X=f(Y), Z=f(Y) }	{}		

Unification Algorithm Examples (II)

- Unify: A = `p(X,f(Y))` and B = `p(a,g(b))`

θ	E	T	S
$\{\}$	$\{$ `p(X,f(Y))=p(a,g(b))` $\}$	`p(X,f(Y))`	`p(a,g(b))`
$\{\}$	$\{$ `X=a`, `f(Y)=g(b)` $\}$	`X`	`a`
$\{$ `X=a` $\}$	$\{$ `f(Y)=g(b)` $\}$	`f(Y)`	`g(b)`
fail			

- Unify: A = `p(X,f(X))` and B = `p(Z,Z)`

θ	E	T	S
$\{\}$	$\{$ `p(X,f(X))=p(Z,Z)` $\}$	`p(X,f(X))`	`p(Z,Z)`
$\{\}$	$\{$ `X=Z`, `f(X)=Z` $\}$	`X`	`Z`
$\{$ `X=Z` $\}$	$\{$ `f(Z)=Z` $\}$	`f(Z)`	`Z`
fail			

A (Schematic) Interpreter for Logic Programs (SLD–resolution)

Input: A logic program P, a query Q

Output: $Q\mu$ (answer substitution) if Q is provable from P, *failure* otherwise

Algorithm:

1. Initialize the "resolvent" R to be $\{Q\}$
2. While R is nonempty do:
   - 2.1. Take the leftmost literal A in R
   - 2.2. Choose a (*renamed*) clause $A' \leftarrow B_1, \ldots, B_n$ from P,
     such that A and A' *unify* with unifier θ
     (if no such clause can be found, branch is *failed*; explore another branch)
   - 2.3. Remove A from R, add $B_1, \ldots, B_n$ to R
   - 2.4. Apply θ to R and Q
3. If R is empty, output Q (a solution). Explore another branch for more sol's.

- Step 2.2 defines *alternative paths* to be explored to find answer(s);
  execution explores this tree (for example, breadth-first).

A (Schematic) Interpreter for Logic Programs (Contd.)

- Since step 2.2 is left open, a given logic *programming* system must specify how it deals with this by providing one (or more)
  - **Search rule(s):** “how are clauses/branches selected in 2.2.”
- If the search rule is not specified execution can be *nondeterministic*, since choosing a different clause (in step 2.2) could lead to different solutions (finding solutions in a different order).

  Example (two valid executions):

`?- pet(X).`	`?- pet(X).`
`X = spot ? ;`	`X = barry ? ;`
`X = barry ? ;`	`X = spot ? ;`
`no`	`no`
`?-`	`?-`

- In fact, there is also some freedom in step 2.1, i.e., a system may also specify:
  - **Computation rule(s):** “how are literals selected in 2.1.”

Running programs

```
C1: pet(X) :- animal(X), barks(X).
C2: pet(X) :- animal(X), meows(X).

C3: animal(spot).
C4: animal(barry).
C5: animal(hobbes).

C6: barks(spot).
C7: meows(barry).
C8: roars(hobbes).
```

- `:- pet(P).`

Q	R	**Clause**	θ
`pet(P)`	*`pet(P)`*	C_2*	$\{P = X_1\}$
`pet(X1)`	*`animal(X1)`*, `meows(X1)`	C_4*	$\{X_1 = \mathrm{barry}\}$
`pet(barry)`	*`meows(barry)`*	C_7	$\{\}$
`pet(barry)`	—	—	—

* means there is a *choice-point*, i.e., there are other clauses whose head unifies.

- System response: `P = barry ?`
- If we type "`;`" after the `?` prompt (i.e., we ask for another solution) the system can go and execute a different branch (i.e., a different choice in C_2* or C_4*).

Running programs (different strategy)

```
C1:  pet(X) :- animal(X), barks(X).
C2:  pet(X) :- animal(X), meows(X).

C3:  animal(spot).
C4:  animal(barry).
C5:  animal(hobbes).

C6:  barks(spot).
C7:  meows(barry).
C8:  roars(hobbes).
```

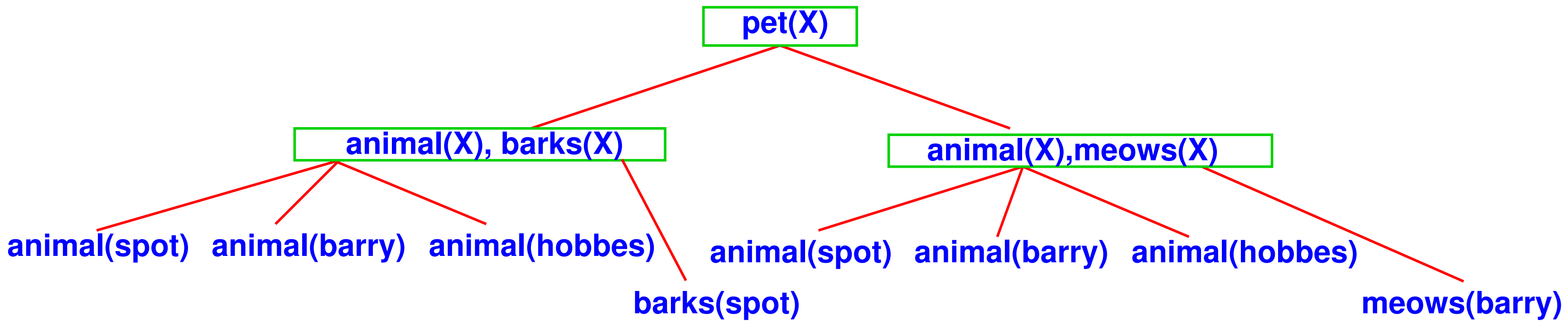
- `:- pet(P).` (different strategy)

Q	R	**Clause**	θ
`pet(P)`	*pet(P)*	C_1*	$\{P = X_1\}$
`pet(X1)`	*animal(X1)*, `barks(X1)`	C_5*	$\{X_1 = \text{hobbes}\}$
`pet(hobbes)`	*barks(hobbes)*	???	*failure*
→ explore another branch (different choice in C_1* or C_5*) to find a solution. We take C_3 instead of C_5:			
`pet(P)`	*pet(P)*	C_1*	$\{P = X_1\}$
`pet(X1)`	*animal(X1)*, `barks(X1)`	C_3*	$\{X_1 = \text{spot}\}$
`pet(spot)`	*barks(spot)*	C_6	$\{\}$
`pet(spot)`	—	—	—

The Search Tree

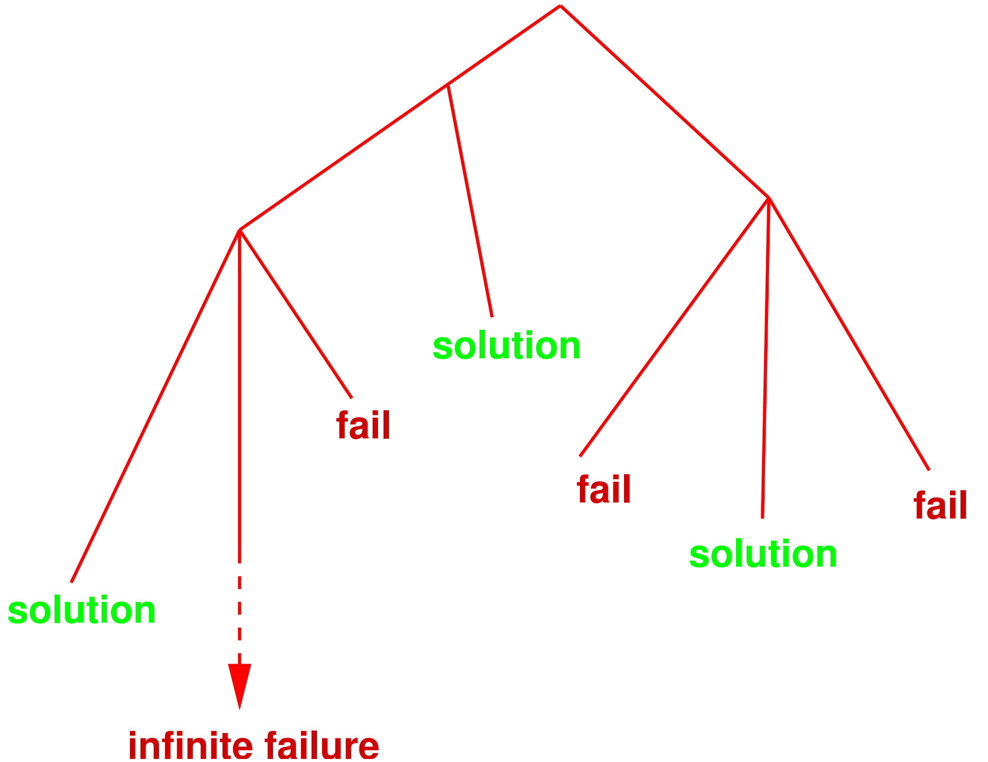
- A query + a logic program together specify a *search tree*.

  Example: query `:- pet(X)` with the previous program generates this search tree (the boxes represent the “and” parts [except leaves]):



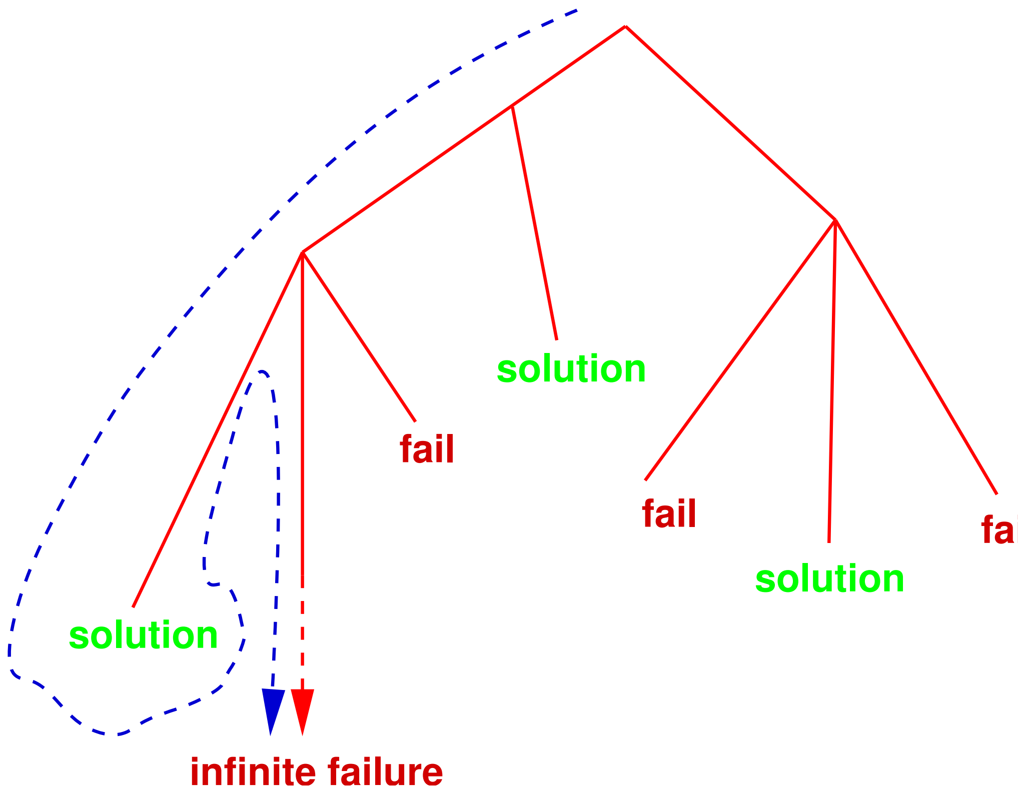
- Different query $\rightarrow$ different tree.
- The search and computation rules explain how the search tree will be explored during execution.
- How can we achieve completeness (guarantee that all solutions will be found)?

Characterization of The Search Tree



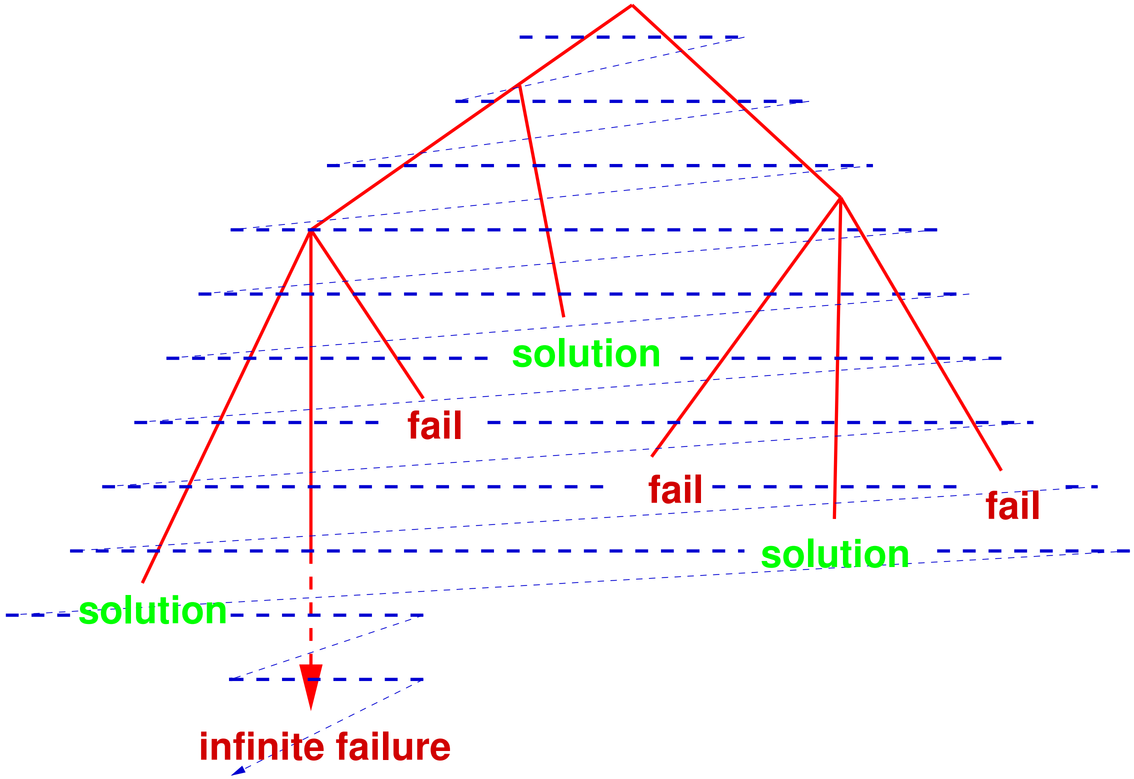
- All solutions are at *finite depth* in the tree.
- Failures can be at finite depth or, in some cases, be an infinite branch.

Depth-First Search



- Incomplete: may fall through an infinite branch before finding all solutions.
- But very efficient: it can be implemented with a call stack, very similar to a traditional programming language.

Breadth-First Search



- Will find all solutions before falling through an infinite branch.
- But costly in terms of time and memory.
- Used in all the following examples (via Ciao's `bf` package).

Selecting breadth-first or depth-first search

- In the Ciao system we can select the search rule using the *packages* mechanism.

- Files should start with the following line:
  - ◇ To execute in *breadth-first* mode:

```
:- module(_,_,['bf/bfall']).
```

  - ◇ To execute in *depth-first* mode:

```
:- module(_,_,[]).
```

See the part on Developing Programs with a Logic Programming System
for more details on the particular system used in the course (Ciao).

Role of Unification in Execution

- As mentioned before, unification used to *access data* and *give values to variables*.

  Example: Consider query `:- animal(A), named(A,Name).` with:

  ```
  animal(dog(barry)).     named(dog(Name),Name).
  ```

- Also, unification is used to *pass parameters* in procedure calls and to *return values* upon procedure exit.

Q	R	**Clause**	θ
`pet(P)`	`pet(P)`	C_1*	{ $P=X_1$ }
`pet(`X_1`)`	`animal(`X_1`), barks(`X_1`)`	C_3*	{ X_1=`spot` }
`pet(spot)`	`barks(spot)`	C_6	{}
`pet(spot)`	—	—	—

“Modes”

- In fact, argument positions are not fixed a priory to be input or output.

  Example: Consider query `:- pet(spot).` vs. `:- pet(X).`

```
or    :- plus( s(0),  s(s(0)), Z).             % Adds
vs.   :- plus( s(0),  Y,         s(s(s(0)))).  % Subtracts
```

- Thus, procedures can be used in different **modes**
  s.t. different sets of arguments are input or output in each mode.

- We sometimes use `+` and `-` to refer to, respectively, and argument being an input or an an output, e.g.:

  `plus(+X, +Y, -Z)` means we call `plus` with

  - ◇ `X` instantiated,
  - ◇ `Y` instantiated, and
  - ◇ `Z` free.

Database Programming

- A Logic Database is a set of facts and rules (i.e., a logic program):

```
father_of(john,peter).
father_of(john,mary).
father_of(peter,michael).

mother_of(mary, david).
```

```
grandfather_of(L,M) :- father_of(L,N),
                       father_of(N,M).
grandfather_of(X,Y) :- father_of(X,Z),
                       mother_of(Z,Y).
```

- Given such database, a logic programming system can answer questions (queries) such as:

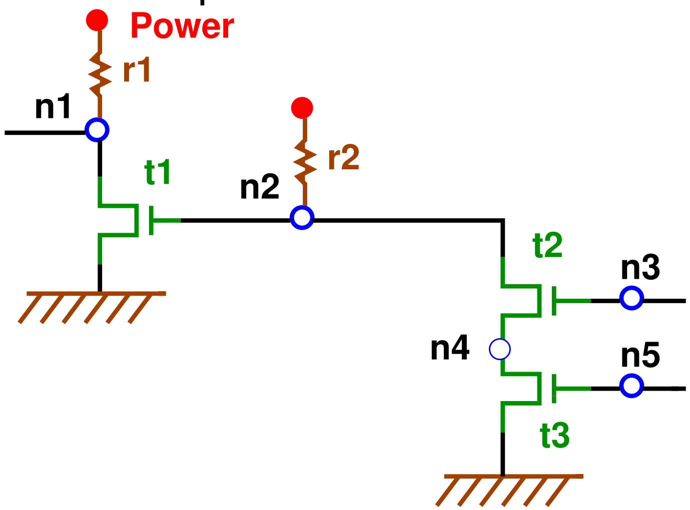
```
?- father_of(john, peter).
yes
?- father_of(john, david).
no
?- father_of(john, X).
X = peter ;
X = mary
```

```
?- grandfather_of(X, michael).
X = john
?- grandfather_of(X, Y).
X = john, Y = michael ;
X = john, Y = david
?- grandfather_of(X, X).
no
```

- Rules for `grandmother_of(X,Y)`?

Database Programming (Contd.)

- Another example:




```
resistor(power,n1).
resistor(power,n2).

transistor(n2,ground,n1).
transistor(n3,n4,n2).
transistor(n5,ground,n4).
```

```
inverter(Input,Output) :-
   transistor(Input,ground,Output), resistor(power,Output).
nand_gate(Input1,Input2,Output) :-
   transistor(Input1,X,Output), transistor(Input2,ground,X),
   resistor(power,Output).
and_gate(Input1,Input2,Output) :-
   nand_gate(Input1,Input2,X), inverter(X, Output).
```

- Query `and_gate(In1,In2,Out)` has solution: `In1=n3, In2=n5, Out=n1`

Structured Data and Data Abstraction (and the '=' Predicate)

- *Data structures* are created using (complex) terms.
- Structuring data is important:

```
course(complog,wed,18,30,20,30,'M.','Hermenegildo',new,5102).
```

- When is the Computational Logic course?

```
?- course(complog,Day,StartH,StartM,FinishH,FinishM,C,D,E,F).
```

- Structured version:

```
    course(complog,Time,Lecturer, Location) :-
        Time = t(wed,18:30,20:30),
        Lecturer = lect('M.','Hermenegildo'),
        Location = loc(new,5102).
```

Note: "`X=Y`" is equivalent to "`'='(X,Y)`"
where the predicate `=/2` is defined as the fact "`'='(X,X).`" – Plain unification!

- Equivalent to:

```
course(complog,  t(wed,18:30,20:30),
       lect('M.','Hermenegildo'),  loc(new,5102)).
```

Structured Data and Data Abstraction (and The Anonymous Variable)

- Given:

```
course(complog,Time,Lecturer, Location) :-
        Time = t(wed,18:30,20:30),
        Lecturer = lect('M.','Hermenegildo'),
        Location = loc(new,5102).
```

- When is the Computational Logic course?

```
?- course(complog, Time, A, B).
```

  has solution:

```
Time=t(wed,18:30,20:30), A=lect('M.','Hermenegildo'), B=loc(new,5102)
```

- Using the *anonymous variable* ("_"):

```
:- course(complog,Time, _, _).
```

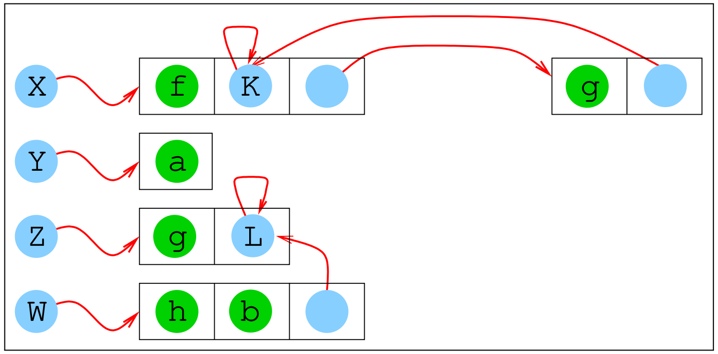
  has solution:

```
Time=t(wed,18:30,20:30)
```

Terms as Data Structures with Pointers

- `main` below is a procedure, that:
  - ◇ creates some data structures, with *pointers* and *aliasing*.
  - ◇ *calls* other *procedures*, *passing* to them *pointers* to these structures.

```
main :-
    X=f(K,g(K)),
    Y=a,
    Z=g(L),
    W=h(b,L),
    % Heap memory at this point ⟶
    p(X,Y),
    q(Y,Z),
    r(W).
```



- Terms are data structures with pointers.
- Logical variables are *declarative* pointers.
  - ◇ Declarative: they can only be assigned once.

Structured Data and Data Abstraction (Contd.)

- The circuit example revisited:

```
resistor(r1,power,n1).       transistor(t1,n2,ground,n1).
resistor(r2,power,n2).       transistor(t2,n3,n4,n2).
                             transistor(t3,n5,ground,n4).
inverter(inv(T,R),Input,Output) :-
  transistor(T,Input,ground,Output),
  resistor(R,power,Output).

nand_gate(nand(T1,T2,R),Input1,Input2,Output) :-
  transistor(T1,Input1,X,Output),
  transistor(T2,Input2,ground,X),
  resistor(R,power,Output).

and_gate(and(N,I),Input1,Input2,Output) :-
   nand_gate(N,Input1,Input2,X), inverter(I,X,Output).
```

- The query `:- and_gate(G,In1,In2,Out).`
  has solution: `G=and(nand(t2,t3,r2),inv(t1,r1)),In1=n3,In2=n5,Out=n1`

Logic Programs and the Relational DB Model

Relational Database		**Logic Programming**
Relation Name	$\rightarrow$	Predicate symbol
Relation	$\rightarrow$	Procedure consisting of ground facts (facts without variables)
Tuple	$\rightarrow$	Ground fact
Attribute	$\rightarrow$	Argument of predicate

Name	Age	Sex
Brown	20	M
Jones	21	F
Smith	36	M

“Person”

```
person(brown,20,male).
person(jones,21,female).
person(smith,36,male).
```

Name	Town	Years
Brown	London	15
Brown	York	5
Jones	Paris	21
Smith	Brussels	15
Smith	Santander	5

“Lived in”

```
lived_in(brown, london,   15).
lived_in(brown, york,      5).
lived_in(jones, paris,    21).
lived_in(smith, brussels,15).
lived_in(smith, santander,5).
```

Logic Programs and the Relational DB Model (Contd.)

- The operations of the relational model are easily implemented as rules.
  - ◇ *Union*: $\texttt{r\_union\_s}(X_1,\ldots,X_n) \leftarrow \texttt{r}(X_1,\ldots,X_n).$

    $\texttt{r\_union\_s}(X_1,\ldots,X_n) \leftarrow \texttt{s}(X_1,\ldots,X_n).$
  - ◇ *Set Difference*: $\texttt{r\_diff\_s}(X_1,\ldots,X_n) \leftarrow \texttt{r}(X_1,\ldots,X_n),\ \textit{not}\ \texttt{s}(X_1,\ldots,X_n).$

    $\texttt{r\_diff\_s}(X_1,\ldots,X_n) \leftarrow \texttt{s}(X_1,\ldots,X_n),\ \textit{not}\ \texttt{r}(X_1,\ldots,X_n).$

    (we postpone the discussion on *negation* until later.)
  - ◇ *Cartesian Product*:

    $\texttt{r\_X\_s}(X_1,\ldots,X_m,X_{m+1},\ldots,X_{m+n}) \leftarrow \texttt{r}(X_1,\ldots,X_m),\texttt{s}(X_{m+1},\ldots,X_{m+n}).$
  - ◇ *Projection*: $\texttt{r13}(X_1,X_3) \leftarrow \texttt{r}(X_1,X_2,X_3).$
  - ◇ *Selection*: $\texttt{r\_selected}(X_1,X_2,X_3) \leftarrow \texttt{r}(X_1,X_2,X_3),\leq(X_2,X_3).$

    (see later for definition of $\leq/2$)
- Derived operations – some can be expressed more directly in LP:
  - ◇ Intersection: $\texttt{r\_meet\_s}(X_1,\ldots,X_n) \leftarrow \texttt{r}(X_1,\ldots,X_n),\ \texttt{s}(X_1,\ldots,X_n).$
  - ◇ Join: $\texttt{r\_joinX2\_s}(X_1,\ldots,X_n) \leftarrow \texttt{r}(X_1,X_2,X_3,\ldots,X_n),\ \texttt{s}(X'_1,X_2,X'_3,\ldots,X'_n).$
- Duplicates an issue: see “setof” later in Prolog.

Deductive Databases

- The subject of “deductive databases” uses these ideas to develop *logic-based databases*.
  - ◇ Often syntactic restrictions (a subset of definite programs) used (e.g. “Datalog” – no functors, no existential variables).
  - ◇ Variations of a “bottom-up” execution strategy used: Use the T_p operator (explained in the theory part) to compute the model, restrict to the query.
  - ◇ Powerful notions of negation supported: S-models
    $\rightarrow$ **Answer Set Programming** (ASP)
    $\rightarrow$ powerful knowledge representation and reasoning systems.

Recursive Programming

- Example: ancestors.

```
parent(X,Y) :- father(X,Y).
parent(X,Y) :- mother(X,Y).

ancestor(X,Y) :- parent(X,Y).
ancestor(X,Y) :- parent(X,Z), parent(Z,Y).
ancestor(X,Y) :- parent(X,Z), parent(Z,W), parent(W,Y).
ancestor(X,Y) :- parent(X,Z), parent(Z,W), parent(W,K), parent(K,Y).
...
```

- Defining ancestor recursively:

```
parent(X,Y) :- father(X,Y).
parent(X,Y) :- mother(X,Y).

ancestor(X,Y) :- parent(X,Y).
ancestor(X,Y) :- parent(X,Z), ancestor(Z,Y).
```

- Exercise: define “related”, “cousin”, “same generation”, etc.

Types

- *Type:* a (possibly infinite) set of terms.
- *Type definition:* A program defining a type.
- *Example:* Weekday:
  - ◇ Set of terms to represent: `'Monday'`, `'Tuesday'`, `'Wednesday'`, ...
  - ◇ Type definition:

```
weekday('Monday').
weekday('Tuesday').  ...
```

- *Example:* Date (weekday * day in the month):
  - ◇ Set of terms to represent: `date('Monday',23)`, `date('Tuesday',24)`, ...
  - ◇ Type definition:

```
date(date(W,D)) :- weekday(W), day_of_month(D).
day_of_month(1).
day_of_month(2).
...
day_of_month(31).
```

Recursive Programming: Recursive Types

- *Recursive types*: defined by recursive logic programs.
- *Example:* natural numbers (simplest recursive data type):
  - ◇ Set of terms to represent: `0, s(0), s(s(0)), ...`
  - ◇ Type definition:

```
nat(0).
nat(s(X)) :- nat(X).
```

  A *minimal recursive predicate*:
  one unit clause and one recursive clause (with a single body literal).
- Types are *runnable* and can be used to check or produce values:
  - ◇ `?- nat(X)` $\Rightarrow$ `X=0; X=s(0); X=s(s(0));` ...
- We can reason about *complexity*, for a given *class of queries* ("*mode*").
  E.g., for mode `nat(`*ground*`)` complexity is *linear* in size of number.
- *Example:* integers:
  - ◇ Set of terms to represent: `0`, `s(0)`, `-s(0)`,...
  - ◇ Type definition:

```
integer( X) :- nat(X).
integer(-X) :- nat(X).
```

Recursive Programming: Arithmetic

- Defining the natural order ($\leq$) of natural numbers:

```
less_or_equal(0,X) :- nat(X).
less_or_equal(s(X),s(Y)) :- less_or_equal(X,Y).
```

  - Multiple uses (modes):
    `less_or_equal(s(0),s(s(0))), less_or_equal(X,0),` ...
  - Multiple solutions:
    `less_or_equal(X,s(0)), less_or_equal(s(s(0)),Y), etc.`

- Addition:

```
plus(0,X,X) :- nat(X).
plus(s(X),Y,s(Z)) :- plus(X,Y,Z).
```

  - Multiple uses (modes): `plus(s(s(0)),s(0),Z), plus(s(s(0)),Y,s(0))`
  - Multiple solutions: `plus(X,Y,s(s(s(0))))`, etc.

Recursive Programming: Arithmetic

- Another possible definition of addition:

```
plus(X,0,X) :- nat(X).
plus(X,s(Y),s(Z)) :- plus(X,Y,Z).
```

- The meaning of `plus` is the same if both definitions are combined.
- Not recommended: several proof trees for the same query $\rightarrow$ not efficient, not concise. We look for minimal axiomatizations.
- The art of logic programming: finding compact and computationally efficient formulations!

- Try to define: `times(X,Y,Z)` (Z = X*Y), `exp(N,X,Y)` (Y = X^N), `factorial(N,F)` (F = N!), `minimum(N1,N2,Min)`, ...

Recursive Programming: Arithmetic

- Definition of `mod(X,Y,Z)`
  "`Z` is the remainder from dividing `X` by `Y`"
  $\exists Q s.t.\ X = Y * Q + Z \wedge Z < Y$
  $\Rightarrow$

```
mod(X,Y,Z) :- less(Z, Y), times(Y,Q,W), plus(W,Z,X).

less(0,s(X)) :- nat(X).
less(s(X),s(Y)) :- less(X,Y).
```

- Another possible definition:

```
mod(X,Y,X) :- less(X, Y).
mod(X,Y,Z) :- plus(X1,Y,X), mod(X1,Y,Z).
```

- The second is much more efficient than the first one
  (compare the size of the proof trees).

Recursive Programming: Arithmetic/Functions

- The Ackermann function:

```
ackermann(0,N) = N+1
ackermann(M,0) = ackermann(M-1,1)
ackermann(M,N) = ackermann(M-1,ackermann(M,N-1))
```

- In Peano arithmetic:

```
ackermann(0,N)           = s(N)
ackermann(s(M1),0)       = ackermann(M1,s(0))
ackermann(s(M1),s(N1))   = ackermann(M1,ackermann(s(M1),N1))
```

- Can be defined as:

```
ackermann(0,N,s(N)).
ackermann(s(M1),0,Val)       :-  ackermann(M1,s(0),Val).
ackermann(s(M1),s(N1),Val)   :-  ackermann(s(M1),N1,Val1),
                                 ackermann(M1,Val1,Val).
```

- In general, *functions* can be coded as a predicate with one more argument, which represents the output (and additional syntactic sugar often available).

Recursive Programming: Arithmetic/Functions (Functional Syntax)

- Syntactic support available (see, e.g., the Ciao *fsyntax* and *functional* packages).
- The Ackermann function (Peano) in Ciao's functional Syntax and defining `s` as a prefix operator:

```
:- use_package(functional).
:- op(500,fy,s).

ackermann(  0,    N) := s N.
ackermann(s M,    0) := ackermann(M, s 0).
ackermann(s M, s N) := ackermann(M, ackermann(s M, N) ).
```

- Convenient in other cases – e.g. for defining types:

```
nat(0).
nat(s(X)) :- nat(X).
```

Using special `:=` notation for the "return" (last) the argument:

```
nat := 0.
nat := s(X) :- nat(X).
```

Recursive Programming: Arithmetic/Functions (Funct. Syntax, Contd.)

Moving body call to head using the ˜ notation ("evaluate and replace with result"):

```
nat := 0.
nat := s(~nat).
```

"~" not needed with `funcional` package if inside its own definition:

```
nat := 0.
nat := s(nat).
```

Using an `:- op(500,fy,s).` declaration to define `s` as a *prefix operator*:

```
nat := 0.
nat := s nat.
```

Using "|" (disjunction):

```
nat := 0 | s nat.
```

Which is exactly equivalent to:

```
nat(0).
nat(s(X)) :- nat(X).
```

Recursive Programming: Lists

- Binary structure: first argument is *element*, second argument is *rest* of the list.
- We need:
  - ◇ A constant symbol: we use the *constant* [ ] ($\rightarrow$ denotes the empty list).
  - ◇ A functor of arity 2: traditionally the dot "." (which is overloaded).
- Syntactic sugar: the term .(X,Y) is denoted by [X|Y] (X is the *head*, Y is the *tail*).

Formal object	"Cons pair" syntax	"Element" syntax
`.(a,[])`	`[a|[]]`	`[a]`
`.(a,.(b,[]))`	`[a|[b|[]]]`	`[a,b]`
`.(a,.(b,.(c,[])))`	`[a|[b|[c|[]]]]`	`[a,b,c]`
`.(a,X)`	`[a|X]`	`[a|X]`
`.(a,.(b,X))`	`[a|[b|X]]`	`[a,b|X]`

- Note that:
  `[a,b]` and `[a|X]` unify with $\{X = [b]\}$
  `[a]` and `[a,b|X]` do not unify
  `[a]` and `[a|X]` unify with $\{X = [\,]\}$
  `[]` and `[X]` do not unify

Recursive Programming: Lists (Contd.)

- Type definition (no syntactic sugar):

```
list([]).
list(.(X,Y)) :- list(Y).
```

- Type definition, with some syntactic sugar (`[ ]` notation):

```
list([]).
list([X|Y]) :- list(Y).
```

- Type definition, using also `functional` package:

```
list := [] | [_|list].
```

- “Exploring” the type:

```
?- list(L).
L = [] ? ;
L = [_] ? ;
L = [_,_] ? ;
L = [_,_,_] ?
...
```

Recursive Programming: Lists (Contd.)

- X is a *member* of the list Y:

```
member(a,[a]).       member(b,[b]).       etc.  ⇒ member(X,[X]).
member(a,[a,c]).     member(b,[b,d]).     etc.  ⇒ member(X,[X,Y]).
member(a,[a,c,d]).   member(b,[b,d,l]).   etc.  ⇒ member(X,[X,Y,Z]).
```

⇒ **member(X,[X|Y]) :- list(Y).**

```
member(a,[c,a]),     member(b,[d,b]).     etc.  ⇒ member(X,[Y,X]).
member(a,[c,d,a]).   member(b,[s,t,b]).   etc.  ⇒ member(X,[Y,Z,X]).
```

⇒ **member(X,[Y|Z]) :- member(X,Z).**

- Resulting definition:

```
member(X,[X|Y]) :- list(Y).
member(X,[_|T]) :- member(X,T).
```

- Uses of member(X,Y):
  - ◇ checking whether an element is in a list (member(b,[a,b,c]))
  - ◇ finding an element in a list (member(X,[a,b,c]))
  - ◇ finding a list containing an element (member(a,Y))

Recursive Programming: Lists (Contd.)

- Combining lists and naturals:
  - Computing the length of a list:

```
len([],0).
len([H|T],s(LT)) :- len(T,LT)
```

  - Adding all elements of a list:

```
sumlist([],0).
sumlist([H|T],S) :- sumlist(T,ST), plus(ST,H,S).
```

  - The type of lists of natural numbers:

```
natlist([]).
natlist([H|T]) :- nat(H), natlist(T).
```

or:

```
natlist := [] | [~nat|natlist].
```

Recursive Programming: Lists (Contd.)

- Exercises:
  - ◇ Define: `prefix(X,Y)` (the list `X` is a prefix of the list `Y`), e.g. `prefix([a, b], [a, b, c, d])`
  - ◇ Define: `suffix(X,Y)`, `sublist(X,Y)`, ...

Recursive Programming: Lists (Contd.)

- Concatenation of lists:
  - ◇ Base case:

    `append([],[a],[a]).   append([],[a,b],[a,b]).` *etc.*

    $\Rightarrow$ **`append([],Ys,Ys) :- list(Ys).`**

  - ◇ Rest of cases (first step):

    `append([a],[b],[a,b]).`

    `append([a],[b,c],[a,b,c]).` *etc.*

    $\Rightarrow$ **`append([X],Ys,[X|Ys]) :- list(Ys).`**

    `append([a,b],[c],[a,b,c]).`

    `append([a,b],[c,d],[a,b,c,d]).` *etc.*

    $\Rightarrow$ **`append([X,Z],Ys,[X,Z|Ys]) :- list(Ys).`**

    This is still infinite $\rightarrow$ we need to generalize more.

Recursive Programming: Lists (Contd.)

- Second generalization:

```
append([X],Ys,[X|Ys]) :- list(Ys).
append([X,Z],Ys,[X,Z|Ys]) :- list(Ys).
append([X,Z,W],Ys,[X,Z,W|Ys]) :- list(Ys).
```

$\Rightarrow$ **`append([X|Xs],Ys,[X|Zs]) :- append(Xs,Ys,Zs).`**

- So, we have:

```
append([],Ys,Ys) :- list(Ys).
append([X|Xs],Ys,[X|Zs]) :- append(Xs,Ys,Zs).
```

- Another way of reasoning: thinking inductively.
  - The base case is: `append([],Ys,Ys):-list(Ys).`
  - If we assume that `append(Xs,Ys,Zs)` works for some iteration, then, in the next one, the following holds: `append([X|Xs],Ys,[X|Zs])`.

Recursive Programming: Lists (Contd.)

- Uses of append:
  - ◇ Concatenate two given lists:

```
?- append([a,b,c],[d,e],L).
L = [a,b,c,d,e] ?
```

  - ◇ Find differences between lists:

```
?- append(D,[d,e],[a,b,c,d,e]).
D = [a,b,c] ?
```

  - ◇ Split a list:

```
?- append(A,B,[a,b,c,d,e]).
A = [],
B = [a,b,c,d,e] ? ;
A = [a],
B = [b,c,d,e] ? ;
A = [a,b],
B = [c,d,e] ? ;
A = [a,b,c],
B = [d,e] ?
...
```

Recursive Programming: Lists (Contd.)

- `reverse(Xs,Ys)`: Ys is the list obtained by reversing the elements in the list Xs

  It is clear that we will need to traverse the list Xs

  For each element X of Xs, we must put X at the end of the rest of the Xs list already reversed:

  ```
  reverse([X|Xs],Ys ) :-
      reverse(Xs,Zs),
      append(Zs,[X],Ys).
  ```

  How can we stop?

  ```
  reverse([],[]).
  ```

- As defined, reverse(Xs,Ys) is very inefficient. Another possible definition: (uses an *accumulating parameter*)

  ```
  reverse(Xs,Ys) :- reverse(Xs,[],Ys).

  reverse([],Ys,Ys).
  reverse([X|Xs],Acc,Ys) :- reverse(Xs,[X|Acc],Ys).
  ```

⇒ Find the differences in terms of efficiency between the two definitions.

Recursive Programming: Binary Trees

- Represented by a ternary functor `tree(Element,Left,Right)`.
- Empty tree represented by `void`.
- Definition:

```
binary_tree(void).
binary_tree(tree(Element,Left,Right)) :-
    binary_tree(Left),
    binary_tree(Right).
```

- Defining `tree_member(Element,Tree)`:

```
tree_member(X,tree(X,Left,Right)) :-
    binary_tree(Left),
    binary_tree(Right).
tree_member(X,tree(Y,Left,Right)) :- tree_member(X,Left).
tree_member(X,tree(Y,Left,Right)) :- tree_member(X,Right).
```

Recursive Programming: Binary Trees

- Defining `pre_order(Tree,Elements)`:
  `Elements` is a list containing the elements of `Tree` traversed in *preorder*.

```
pre_order(void,[]).
pre_order(tree(X,Left,Right),Elements) :-
     pre_order(Left,ElementsLeft),
     pre_order(Right,ElementsRight),
     append([X|ElementsLeft],ElementsRight,Elements).
```

- Exercise – define:
  - ◇ `in_order(Tree,Elements)`
  - ◇ `post_order(Tree,Elements)`

Polymorphism

- Note that the two definitions of `member/2` can be used *simultaneously*:

```
lt_member(X,[X|Y]) :- list(Y).
lt_member(X,[_|T]) :- lt_member(X,T).

lt_member(X,tree(X,L,R)) :- binary_tree(L), binary_tree(R).
lt_member(X,tree(Y,L,R)) :- lt_member(X,L).
lt_member(X,tree(Y,L,R)) :- lt_member(X,R).
```

  Lists only unify with the first two clauses, trees with clauses 3–5!

- `:- lt_member(X,[b,a,c]).`
  `X = b ; X = a ; X = c`

- `:- lt_member(X,tree(b,tree(a,void,void),tree(c,void,void))).`
  `X = b ; X = a ; X = c`

- Also, try (somewat surprising): `:- lt_member(M,T).`

Recursive Programming: Manipulating Symbolic Expressions

- Recognizing (and generating!) polynomials in some term X:
  - ◇ X is a polynomial in X
  - ◇ a constant is a polynomial in X
  - ◇ sums, differences and products of polynomials in X are polynomials
  - ◇ also polynomials raised to the power of a natural number and the quotient of a polynomial by a constant

```
polynomial(X,X).
polynomial(Term,X)          :- pconstant(Term).
polynomial(Term1+Term2,X) :- polynomial(Term1,X), polynomial(Term2,X).
polynomial(Term1-Term2,X) :- polynomial(Term1,X), polynomial(Term2,X).
polynomial(Term1*Term2,X) :- polynomial(Term1,X), polynomial(Term2,X).
polynomial(Term1/Term2,X) :- polynomial(Term1,X), pconstant(Term2).
polynomial(Term1^N,X)      :- polynomial(Term1,X), nat(N).
```

Recursive Programming: Manipulating Symb. Expressions (Contd.)

- Symbolic differentiation: `deriv(Expression, X, Derivative)`

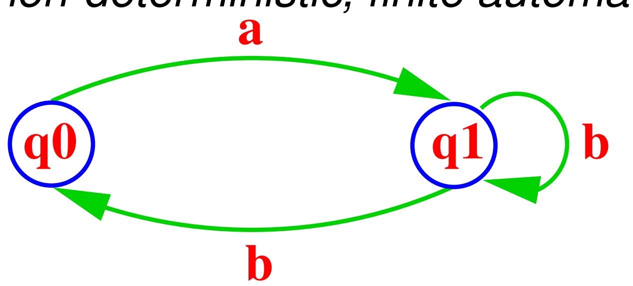
```
deriv(X,X,s(0)).
deriv(C,X,0)                      :- pconstant(C).
deriv(U+V,X,DU+DV)                :- deriv(U,X,DU), deriv(V,X,DV).
deriv(U-V,X,DU-DV)                :- deriv(U,X,DU), deriv(V,X,DV).
deriv(U*V,X,DU*V+U*DV)            :- deriv(U,X,DU), deriv(V,X,DV).
deriv(U/V,X,(DU*V-U*DV)/V^s(s(0))) :- deriv(U,X,DU), deriv(V,X,DV).
deriv(U^s(N),X,s(N)*U^N*DU)       :- deriv(U,X,DU), nat(N).
deriv(log(U),X,DU/U)              :- deriv(U,X,DU).
...
```

- `?- deriv(s(s(s(0)))*x+s(s(0)),x,Y)`.

- A simplification step can be added.

Recursive Programming: Automata (Graphs)

- Recognizing the sequence of characters accepted by the following *non-deterministic, finite automaton* (NDFA):



where **q0** is both the *initial* and the *final* state.

- Strings are represented as lists of constants (e.g., `[a,b,b]`).
- Program:

```
initial(q0).        delta(q0,a,q1).
                    delta(q1,b,q0).
final(q0).          delta(q1,b,q1).

accept(S)          :- initial(Q), accept_from(S,Q).

accept_from([],Q)      :- final(Q).
accept_from([X|Xs],Q) :- delta(Q,X,NewQ), accept_from(Xs,NewQ).
```

Recursive Programming: Automata (Graphs) (Contd.)

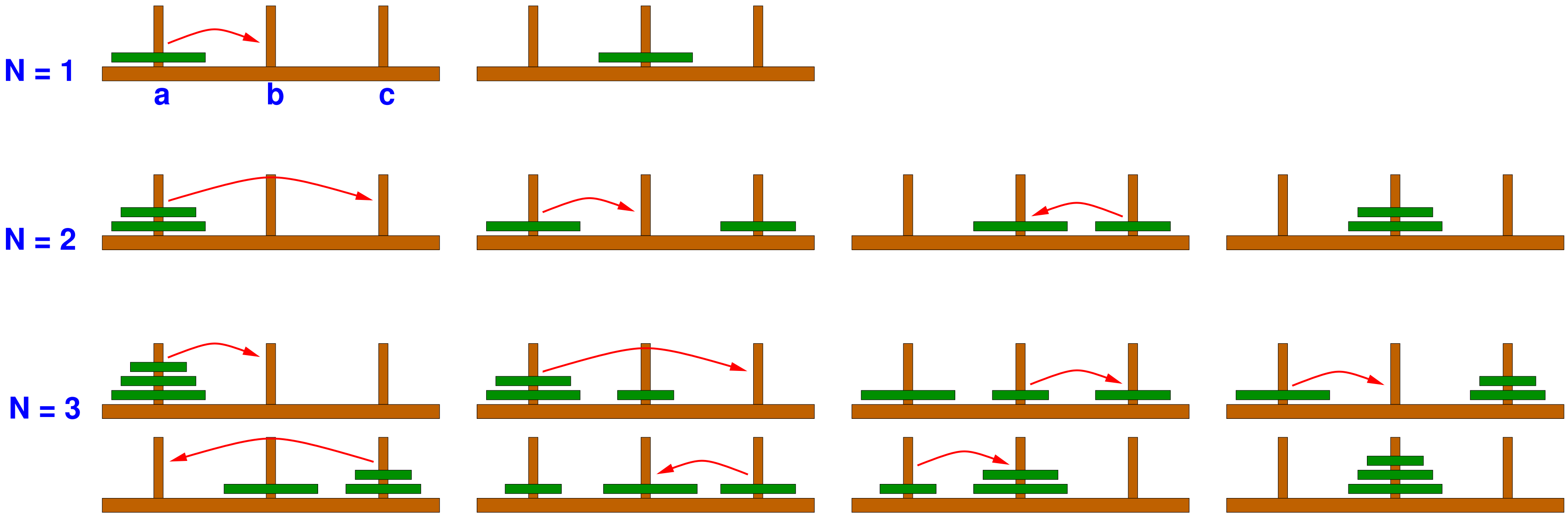
- A *nondeterministic, stack, finite automaton* (NDSFA):

```
accept(S) :- initial(Q), accept_from(S,Q,[]).

accept_from([],Q,[])     :- final(Q).
accept_from([X|Xs],Q,S) :- delta(Q,X,S,NewQ,NewS),
                           accept_from(Xs,NewQ,NewS).

initial(q0).
final(q1).

delta(q0,X,Xs,q0,[X|Xs]).
delta(q0,X,Xs,q1,[X|Xs]).
delta(q0,X,Xs,q1,Xs).
delta(q1,X,[X|Xs],q1,Xs).
```

- What sequence does it recognize?

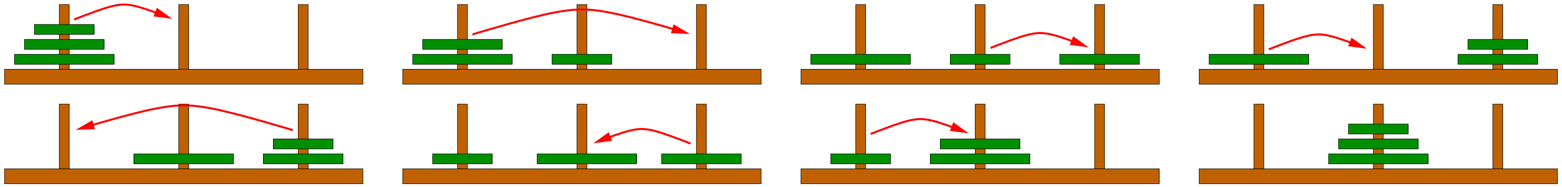
Recursive Programming: Towers of Hanoi

- *Objective:*
  - ◇ Move tower of N disks from peg a to peg b, with the help of peg c.
- *Rules:*
  - ◇ Only one disk can be moved at a time.
  - ◇ A larger disk can never be placed on top of a smaller disk.



Recursive Programming: Towers of Hanoi (Contd.)

- We will call the main predicate `hanoi_moves(N,Moves)`
- `N` is the number of disks and `Moves` the corresponding list of "moves".
- Each move `move(A, B)` represents that the top disk in A should be moved to B.
- *Example:*

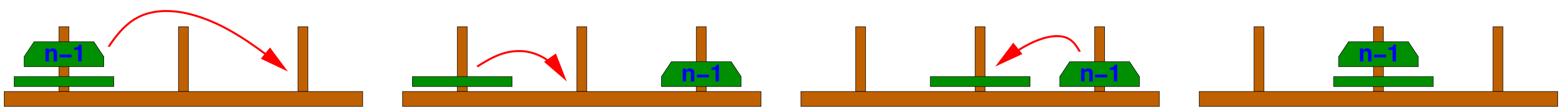


is represented by:

```
hanoi_moves( s(s(s(0))),
             [ move(a,b), move(a,c), move(b,c), move(a,b),
               move(c,a), move(c,b), move(a,b) ])
```

Recursive Programming: Towers of Hanoi (Contd.)

- A general rule:



- We capture this in a predicate `hanoi(N,Orig,Dest,Help,Moves)` where "`Moves` contains the moves needed to move a tower of `N` disks from peg `Orig` to peg `Dest`, with the help of peg `Help`."

```
hanoi(s(0),Orig,Dest,_Help,[move(Orig, Dest)]).
hanoi(s(N),Orig,Dest,Help,Moves) :-
        hanoi(N,Orig,Help,Dest,Moves1),
        hanoi(N,Help,Dest,Orig,Moves2),
        append(Moves1,[move(Orig, Dest)|Moves2],Moves).
```

- And we simply call this predicate:

```
hanoi_moves(N,Moves) :-
        hanoi(N,a,b,c,Moves).
```

Learning to Compose Recursive Programs

- To some extent it is a simple question of practice.
- By generalization (as in the previous examples): elegant, but sometimes difficult? (Not the way most people do it.)
- Think inductively: state first the base case(s), and then think about the general recursive case(s).
- Sometimes it may help to compose programs with a given use in mind (e.g., “forwards execution”), making sure it is declaratively correct. Consider then also if alternative uses make sense.
- Sometimes it helps to look at well-written examples and use the same “schemas.”
- Using a global top-down design approach can help (in general, not just for recursive programs):
  - State the general problem.
  - Break it down into subproblems.
  - Solve the pieces.
- Again, the best approach: practice, practice, practice.