Model Checking for Process Rewrite Systems and a Class of Action-Based Regular Properties

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Abstract. We consider the model checking problem for Process Rewrite Systems (*PRSs*), an infinite-state formalism (non Turing-powerful) which subsumes many common models such as Pushdown Processes and Petri Nets. *PRSs* can be adopted as formal models for programs with dynamic creation and synchronization of concurrent processes, and with recursive procedures. The model-checking problem for *PRSs* w.r.t. action-based linear temporal logic (*ALTL*) is undecidable. However, decidability for some interesting fragment of *ALTL* remains an open question. In this paper we state decidability results concerning generalized acceptance properties about infinite derivations (infinite term rewriting) in *PRSs*. As a consequence, we obtain decidability of the model-checking (restricted to infinite runs) for *PRSs* and a meaningful fragment of *ALTL*.

1 Introduction

Automatic verification of systems is nowadays one of the most investigated topics. A major difficulty to face when considering this problem is that reasoning about systems in general may require dealing with infinite–state models. Software systems may introduce infinite states both manipulating data ranging over infinite domains, and having unbounded control structures such as recursive procedure calls and/or dynamic creation of concurrent processes (e.g. multi–threading). Many different formalisms have been proposed for the description of infinite–state systems. Among the most popular are the well known formalisms of Context Free Processes, Pushdown Processes, Petri Nets, and Process Algebras. The first two are models of sequential computation, whereas Petri Nets and Process Algebra explicitly take into account concurrency. The model checking problem for these infinite–state formalisms have been studied in the literature. As far as Context Free Processes and Pushdown Processes are concerned, decidability of the modal μ–calculus, the most powerful of the modal and temporal logics used for verification, has been established (see [2, 5, 10, 14]). In [4, 8, 9], model checking for Petri nets has been studied. The branching temporal logic as well as the state-based linear temporal logic are undecidable even for restricted logics. Fortunately, the model checking for action-based linear temporal logic (*ALTL*) [8, 9] is decidable.

Verification of formalisms which accommodate both parallelism and recursion is a challenging problem. In order to formally study this kind of systems,

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recently the formal framework of Process Rewrite Systems (*PRSs*) has been introduced [12, 13]. This framework (non Turing-powerful), which is based on term rewriting, subsumes many common infinite–states models such as Pushdown Processes and Petri Nets. *PRSs* can be adopted as formal models for programs with dynamic creation and (a restricted form of) synchronization of concurrent processes, and with recursive procedures. The decidability results already known in the literature for the general framework of *PRSs* concern the reachability problem between two fixed terms and the *reachable property* problem [12, 13]. This last is the problem of deciding whether there is a reachable term that satisfies certain properties that can be encoded as follows: some given rewrite rules are applicable and/or other given rewrite rules are *not* applicable. Decidability of this problem can be also used to decide the deadlock reachability problem. Recently, in [3], symbolic reachability analysis is investigated (i.e., the constructibility problem of the potentially infinite set of terms that are reachable from a given possibly infinite set of terms). However, the algorithm given in [3] can be applied only to a subclass of *PRSs* (strictly less expressive), i.e, the *synchronization–free* PRSs (the so–called PAD systems) which subsume Pushdown processes and the *synchronization–free* Petri nets. As concerns the *ALTL* model–checking problem, it is undecidable for the whole class of *PRSs* [1, 12, 13]. It remains undecidable even for restricted models such as PA processes [1] (these systems correspond to a subclass, strictly less expressive, of PAD systems). Fortunately, Bouajjani in [1] proved that for the complement of *simple ALTL*[1] (*simple ALTL* corresponds to Büchi ω-automata where there are only self–loop), model–checking PA processes is decidable. Anyway, decidability for some interesting fragment of *ALTL* and the general framework of *PRSs* remains an open question.

In this paper we prove decidability of the model–checking problem (restricted to infinite runs) for the whole class of *PRSs* w.r.t. a meaningful fragment of *ALTL* that captures, exactly, the class of regular properties invariant under permutation of atomic actions (along infinite runs). This fragment (closed under boolean connectives) is defined as follows:

$$\varphi ::= F\,\psi \mid GF\,\psi \mid \neg\varphi \mid \varphi \wedge \varphi \tag{1}$$

where ψ is an *ALTL* propositional formula (i.e, a boolean combination of atomic actions). Within this fragment, class of properties useful in system verification can be expressed: some *safety properties* (e.g., $G\,\psi_1$), some *guarantee properties* (e.g., $F\,\psi_1$), some *obligation properties* (e.g., $F\,\psi_1 \rightarrow F\,\psi_2$, or $G\,\psi_1 \rightarrow G\,\psi_2$), some *recurrence properties* (e.g., $GF\,\psi_1$), some *persistence properties* (e.g., $FG\,\psi_1$), and finally some *reactivity properties* (e.g., $GF\,\psi_1 \rightarrow GF\,\psi_2$) [2].

[1] *Simple ALTL* is *not* closed under negation, and is defined as follows:

$$\varphi ::= \psi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \langle a \rangle \varphi \mid G\,\psi \mid \psi\, U\,\varphi$$

where ψ is an *ALTL* propositional formula, a is an atomic action, and $<a>$, G, and U are the *next*, *always*, and *until* operators.

[2] ψ_1 and ψ_2 denote *ALTL* propositional formulae.

Notice that important classes of properties like invariants, as well as strong and weak fairness constraints, can be expressed. Moreover, notice that this fragment and *simple ALTL* are incomparable (in particular, strong fairness cannot be expressed by *simple ALTL*).

In order to prove our result, we introduce the notion of *Multi Büchi Rewrite System* (*MBRS*) that is, informally speaking, a *PRS* with a finite number of accepting components, where each component is a subset of the *PRS*. Then, we reduce our initial problem to that of verifying the existence of infinite derivations (infinite term rewriting) in *MBRS*s satisfying *generalized* acceptance properties (*a la* Büchi). Finally, we prove decidability of this last problem by a reduction to the *ALTL* model–checking problem for Petri nets and Pushdown processes (that is decidable). There are two main steps in the proof of decidability:

– First, we prove decidability of a problem concerning the existence of *finite* derivations leading to a given term and satisfying generalized acceptance properties. This problem is strictly more general than reachability problem and is not comparable with the reachable property problem of Mayr [12, 13]. Moreover, our approach is substantially different from that used by Mayr.
– The second step concerns reasoning about infinite derivations in *PRSs* which have not been investigated (to the best of our knowledge) in other papers on *PRSs*.

The framework of *MBRS*s, introduced in this paper, can be also used to suitably express other important class of regular properties, for example, the *ALTL* fragment given by *simple ALTL* without the *next* operator. Properties in this fragment can be translated in orderings of occurrences of rules belonging to the accepting components of the given *MBRS*. Actually, we are working on the satisfiability problem of the conjunction of this *ALTL* fragment with the fragment (1) w.r.t. *PRSs* (i.e., the problem about the existence of an infinite run in the given *PRS* satisfying a given formula), using a technique similar to that used in this paper. The result would be particularly interesting, since it is possible to prove (using a result of Emerson [7]) that the *positive* boolean combination of this fragment with the fragment (1) subsumes the relevant *ALTL* fragment (closed under boolean connectives) with the *always* and *eventually* operators (G and F) nested arbitrarily, i.e., (linear-time) Lamport logic[3]. This means that the main result of this paper is an intermediate but fundamental step for resolving the model–checking problem of *PRSs* against a full action-based temporal logic, i.e., (linear-time) Lamport logic.

Plan of the paper: In Section 2, we recall the framework of Process Rewrite Systems and *ALTL* logic. In Section 3, we introduce the notion of *Multi Büchi Rewrite System*, and show how our decidability result about generalized acceptance properties of infinite derivations in *PRSs* can be used in model-checking for the *ALTL* fragment (1). In Sections 4 and 5, we prove our decidability result.

[3] Since Lamport logic (as well as the fragment (1)) is closed under negation, decidability of the satisfiability problem implies decidability of the model–checking problem, and viceversa.

Finally, in Section 6, we conclude with some considerations about the complexity of the considered problem.

Several proofs are omitted for lack of space. They can be found in the longer version of this paper that can be requested to the author.

2 Preliminaries

2.1 Process Rewrite Systems

Let $Var = \{X, Y, \ldots\}$ be a finite set of *process variables*. The set T of *process terms* t over Var is defined by the following syntax:

$$t ::= \varepsilon \mid X \mid t.t \mid t \| t$$

where $X \in Var$, ε is the empty term, and "$\|$" (resp., ".") denotes parallel composition (resp., sequential composition). We always work with equivalences classes of process terms modulo commutativity and associativity of "$\|$", and modulo associativity of ".". Moreover, ε will act as the identity for both parallel and sequential composition.

Definition 1 ([13]). *A* Process Rewrite System (PRS *for short*) *over* Var *and an alphabet of atomic actions* Σ *is a* finite *set of rewrite rules of the form* $t \xrightarrow{a} t'$*, where* t $(\neq \varepsilon)$ *and* t' *are terms in* T*, and* $a \in \Sigma$.

A *PRS* $\Re$ induces a Labelled Transition System (LTS) with set of states T, and a transition relation $\rightarrow \subseteq T \times \Sigma \times T$ defined by the following inference rules:

$$\frac{(t \xrightarrow{a} t') \in \Re}{t \xrightarrow{a} t'} \qquad \frac{t_1 \xrightarrow{a} t_1'}{t_1 \| t \xrightarrow{a} t_1' \| t} \qquad \frac{t_1 \xrightarrow{a} t_1'}{t_1.t \xrightarrow{a} t_1'.t}$$

where t, t', t_1, t_1' are process terms and $a \in \Sigma$. In similar way we define for every rule $r \in \Re$ the notion of *one–step derivation* relation by r, denoted by $\overset{r}{\Rightarrow}_{\Re}$.

A *path in* $\Re$ from $t \in T$ is a (finite or infinite) sequence $\pi = t_0 \xrightarrow{a_0} t_1 \xrightarrow{a_1} t_2 \ldots$ such that $t = t_0$ and every triple $t_i \xrightarrow{a_i} t_{i+1}$ is a LTS edge. We write π^i for the path $t_i \xrightarrow{a_i} t_{i+1} \xrightarrow{a_{i+1}} \ldots$. Let *firstact*$(\pi) := a_0$. A *run in* $\Re$ from t is a maximal path from t, i.e., a path that is either infinite, or terminates in a term without successors. We denote by $runs_{\Re}(t)$ (resp., $runs_{\Re,\infty}(t)$) the set of runs (resp., infinite runs) in $\Re$ from t, and by $runs(\Re)$ the set of all the runs in $\Re$.

Given a finite (resp., infinite) sequence $\sigma = r_1 r_2 \ldots$ of rules in $\Re$, a *finite* (resp., *infinite*) *derivation* in $\Re$ from a term t (through σ), is a finite (resp., infinite) sequence d of the form $t_0 \overset{r_1}{\Rightarrow}_{\Re} t_1 \overset{r_2}{\Rightarrow}_{\Re} t_2 \ldots$ such that $t_0 = t$ and every triple $t_i \overset{r_i}{\Rightarrow}_{\Re} t_{i+1}$ is a one–step derivation. If d is finite and terminates in the term t', we say t' is *reachable* in $\Re$ from t (through derivation d). If σ is empty, we say d is a *null derivation*. For terms $t, t' \in T$ and a rule sequence σ, we write $t \overset{\sigma}{\Rightarrow}_{\Re}$ (resp., $t \overset{\sigma}{\Rightarrow}_{\Re} t'$) to mean that there exists a derivation (resp., a finite derivation terminating in t') from t through σ.

For technical reasons, we also consider *PRSs* in a restricted syntactical form called *normal form* [13]. A *PRS* $\Re$ is said to be in *normal form* if every rule $r \in \Re$ has one of the following forms:

PAR rules: $X_1 \| X_2 \ldots \| X_p \xrightarrow{a} Y_1 \| Y_2 \ldots \| Y_q$ where $p \in \mathbb{N} \setminus \{0\}$ and $q \in \mathbb{N}$.
SEQ rules: $X \xrightarrow{a} Y.Z$ or $X.Y \xrightarrow{a} Z$ or $X \xrightarrow{a} Y$ or $X \xrightarrow{a} \varepsilon$.

with $X, Y, Z, X_i, Y_j \in Var$. A *PRS* where all the rules are SEQ (resp., PAR) rules is called *sequential* (resp., *parallel*) *PRS*.

2.2 *ALTL* (Action–Based LTL) and *PRSs*

The set of *ALTL* formulae over a set Σ of atomic actions is defined as follows:

$$\varphi ::= true \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle a \rangle \varphi \mid \varphi U \varphi$$

where $a \in \Sigma$, $\langle a \rangle$ is the *next* operator, and U is the *until* operator. We also define $F\varphi := true U \varphi$ ("*eventually* φ") and its dual $G\varphi := \neg F \neg \varphi$ ("*always* φ"). Given a *PRS* $\Re$ and an *ALTL* formula φ, the set of the runs in $\Re$ *satisfying* φ, in symbols $[[\varphi]]_\Re$, is defined inductively on the structure of φ as follows:

- $[[true]]_\Re = runs(\Re)$,
- $[[\neg\varphi]]_\Re = runs(\Re) \setminus [[\varphi]]_\Re$,
- $[[\varphi_1 \wedge \varphi_2]]_\Re = [[\varphi_1]]_\Re \cap [[\varphi_2]]_\Re$,
- $[[\langle a \rangle \varphi]]_\Re = \{\pi \in runs(\Re) \mid \mathit{firstact}(\pi) = a \text{ and } \pi^1 \in [[\varphi]]_\Re\}$,
- $[[\varphi_1 U \varphi_2]]_\Re = \{\pi \in runs(\Re) \mid \text{for some } i \geq 0 : \pi^i \in [[\varphi_2]]_\Re \text{ and for all } j < i,\ \pi^j \in [[\varphi_1]]_\Re \}$.

For any term $t \in T$, we say t satisfies φ (resp., satisfies φ restricted to infinite runs) w.r.t. $\Re$, in symbols $t \models_\Re \varphi$ (resp., $t \models_{\Re,\infty} \varphi$), if $runs_\Re(t) \subseteq [[\varphi]]_\Re$ (resp., $runs_{\Re,\infty}(t) \subseteq [[\varphi]]_\Re$). The model-checking problem (resp., model–checking problem restricted to infinite runs) for *ALTL* w.r.t. *PRSs* is the problem of deciding whether, given a *PRS* $\Re$, an *ALTL* formula φ and a term $t \in T$, $t \models_\Re \varphi$ (resp., $t \models_{\Re,\infty} \varphi$). The following is a well–known result:

Proposition 1 (see [2, 8, 12]). *The model–checking problem for* ALTL *w.r.t. parallel (resp., sequential)* PRSs*, possibly restricted to infinite runs, is decidable.*

In this paper we are interested in the model-checking problem (restricted to infinite runs) for unrestricted *PRSs* against the following *ALTL* fragment:

$$\varphi ::= F\,\psi \mid GF\,\psi \mid \neg\varphi \mid \varphi \wedge \varphi \tag{1}$$

where ψ denotes an *ALTL propositional* formula defined by the following syntax: $\psi ::= \langle a \rangle\ true \mid \psi \wedge \psi \mid \neg\psi$ (where $a \in \Sigma$).

3 Multi Büchi Rewrite Systems

In order to prove the main result of this paper, i.e. the decidability of the model-checking problem (restricted to infinite runs) of *PRSs* against the *ALTL* fragment defined in Subsection 2.2, we introduce the framework of Multi Büchi Rewrite Systems.

Definition 2. *A* Multi Büchi Rewrite System (MBRS) (*with* n accepting components) *over* Var *and* Σ *is a tuple* $M = \langle \Re, \langle \Re_1, \ldots, \Re_n \rangle \rangle$*, where* $\Re$ *is a* PRS *over* Var *and* Σ*, and, for all* $i = 1, \ldots, n$*,* $\Re_i \subseteq \Re$*.* $\Re$ *is called the* support *of* M*.*

We say M is a *MBRS in normal form* (resp., *sequential MBRS*, *parallel MBRS*) if the support $\Re$ is in normal form (resp., sequential, parallel).

For a rule sequence σ in $\Re$ the *finite acceptance of* σ w.r.t. M, denoted by $\Upsilon_M^f(\sigma)$, is the set $\{i \in \{1, \ldots, n\} | \ \sigma$ contains some occurrence of rule in $\Re_i\}$. The *infinite acceptance of* σ w.r.t. M, denoted by $\Upsilon_M^\infty(\sigma)$, is the set $\{i \in \{1, \ldots, n\} |$ σ contains infinite occurrences of some rule in $\Re_i\}$. Given $K, K_\omega \subseteq \{1, \ldots, n\}$ and a derivation d of the form $t \overset{\sigma}{\Rightarrow}_\Re$, we say d is a $(K, K_\omega)-$*accepting derivation* in M if $\Upsilon_M^f(\sigma) = K$ and $\Upsilon_M^\infty(\sigma) = K_\omega$. Moreover, we say d has *finite acceptance* (resp., *infinite acceptance*) K (resp., K_ω) in M. For all $n \in \mathbb{N} \setminus \{0\}$, let us denote by P_n the set $2^{\{1,\ldots,n\}}$ (i.e., the set of the subsets of $\{1, \ldots, n\}$).
Now, let us consider the following problem:

Fairness Problem: *Given a* MBRS $M = \langle \Re, \langle \Re_1, \ldots, \Re_n \rangle \rangle$ *over Var and* Σ*, a process term* t*, and two sets* $K, K_\omega \in P_n$*, is there a* (K, K_ω)*-accepting infinite derivation in* M *from* t?

Without loss of generality we can assume that the input term t in the Fairness Problem is a process variable. In fact, if $t \notin Var$, then we add a fresh variable X and a rule of the form $X \to t$ whose finite acceptance is the empty set.

As stated by the following Theorem, the Fairness Problem represents a suitable encoding of our initial problem in the framework of *MBRS*s.

Theorem 1. *Model–checking* PRSs *against the considered* ALTL *fragment, restricted to infinite runs, is polynomial-time reducible to the Fairness Problem.*

In the remainder of this paper we prove that the Fairness Problem is decidable. We proceed in two steps. First, in Section 4 we decide the problem for the class of *MBRS*s in normal form. Then, in Section 5 we extend the result to the whole class of *MBRS*s. For the proof we need some preliminary decidability results, stated by the following Propositions 2–4, concerning acceptance properties of derivations in parallel and sequential *MBRS*s. In particular, the problems in Propositions 2–3 (resp., in Proposition 4) are polynomial-time reducible to the *ALTL* model–checking problem for parallel (resp., sequential) *PRSs* that is decidable (see Proposition 1).

Proposition 2. *Given a* parallel MBRS M_P *over* Var *and with* n *accepting components, two variables* $X, Y \in Var$ *and* $K \in P_n$*, it is decidable whether* *there is a finite derivation in M_P of the form $X \overset{\sigma}{\Rightarrow}$ (resp., of the form $X \overset{\sigma}{\Rightarrow} Y$, of the form $X \overset{\sigma}{\Rightarrow} \varepsilon$, of the form $X \overset{\sigma}{\Rightarrow} t\|Y$ with $|\sigma| > 0$) such that $\Upsilon^f_{M_P}(\sigma) = K$.*

Proposition 3. *Let M_{P_1} and M_{P_2} be two* parallel MBRSs *over Var, with the same support $\Re_P$, and with n accepting components. Given $X \in Var$, $K, K_\omega \in P_n$, and a subset Λ of $\Re_P$, it is decidable whether:* (1) *there exists a (K, K_ω)-accepting infinite derivation in M_{P_1} from X;* (2) *there exists a derivation in $\Re_P$ of the form $X \overset{\sigma}{\Rightarrow}$ such that $\Upsilon^f_{M_{P_1}}(\sigma) = K$, $\Upsilon^\infty_{M_{P_1}}(\sigma) \cup \Upsilon^f_{M_{P_2}}(\sigma) = K_\omega$, and σ is either infinite or contains some occurrence of rule in Λ.*

Now, we give the notion of s-reachability in sequential *PRSs*.

Definition 3. *Given a* sequential PRS $\Re_S$ *over Var, and $X, Y \in Var$, Y is* s-reachable from X in $\Re_S$ *if there exists a term t of the form $Y.X_1.X_2.\ldots.X_n$ (where $X_i \in Var$ for any $i = 1, \ldots, n$, and $n \geq 0$) such that $X \Rightarrow t$.*

Proposition 4. *Given a* sequential MBRS M_S *over Var and with n accepting components, two variables $X, Y \in Var$, and two sets $K, K_\omega \in P_n$, it is decidable whether:* (1) *Y is s-reachable from X in M_S through a $(K, \emptyset)$-accepting derivation;* (2) *there is a (K, K_ω)-accepting infinite derivation in M_S from X.*

4 Decidability of the Fairness Problem for *MBRS*s in Normal Form

In this subsection we prove decidability of the Fairness Problem for the class of *MBRS*s in normal form. We fix a *MBRS* in normal form $M = \langle \Re, \langle \Re_1, \ldots, \Re_n \rangle \rangle$ over Var and Σ, and two elements K and K_ω of P_n. Given $X \in Var$, we have to decide if there exists a (K, K_ω)-accepting infinite derivation in M from X.

Remark 1 Since M is in normal form (and in the following we only consider derivations starting from variables or terms in which no sequential composition occurs) we can limit ourselves to consider only *terms in normal form*, defined as $t ::= \varepsilon \mid X \mid t\|t \mid t.X$ (where $X \in Var$). In fact, given a term in normal form t, each term t' reachable from t in M is still in normal form.

There are two main steps for the decidability proof of the Fairness Problem.

Step 1 First, we prove decidability of the following problem:
Problem 1 (Finite Derivations): Given $X, Y \in Var$ and $K' \in P_n$, is there a finite derivation in M of the form $X \overset{\sigma}{\Rightarrow}$ (resp., $X \overset{\sigma}{\Rightarrow} Y$) such that $\Upsilon^f_M(\sigma) = K'$?

Step 2 Using decidability of Problem **1**, we show that the Fairness Problem can be reduced to (a combination of) simpler and decidable problems regarding acceptance properties of derivations of parallel and sequential *MBRS*s.

Before illustrating our approach, we need additional notation.

In the following, $M_P = \langle \Re_P, \langle \Re_{P,1}, \ldots, \Re_{P,n} \rangle \rangle$ denotes the restriction of M to the PAR rules, i.e., $\Re_P$ (resp., $\Re_{P,i}$ for $i = 1, \ldots, n$) is the set $\Re$ (resp., $\Re_i$ for $i = 1, \ldots, n$) restricted to the PAR rules. Moreover, we shall use a fresh variable Z_F, and denote by T (resp., T_{PAR}, T_{SEQ}) the set of process terms in normal form (resp., in which no sequential composition occurs, in which no parallel composition occurs) over $Var \cup \{Z_F\}$.

Definition 4 (Subderivation). *Let* $\overline{t} \overset{\lambda}{\Rightarrow} t \| (s.X) \overset{\sigma}{\Rightarrow}$ *be a derivation*[4] *in* $\Re$ *from* $\overline{t} \in T$. *The set of the* subderivations d' of $d = (t\|(s.X) \overset{\sigma}{\Rightarrow})$ from s *is inductively defined as follows:*

1. *if* d *is a null derivation* or $s = \varepsilon$ or d *is of the form* $t\|(Z.X) \overset{r}{\Rightarrow} t\|Y \overset{\sigma'}{\Rightarrow}$ *(with* $r = Z.X \overset{a}{\rightarrow} Y$ *and* $s = Z$*), then* d' *is the null derivation from* s*;*
2. *if* d *is of the form* $t\|(s.X) \overset{r}{\Rightarrow} t\|(s'.X) \overset{\sigma'}{\Rightarrow}$ *(with* $s \overset{r}{\Rightarrow} s'$ *and* $r \in \Re$*) and* $s' \overset{\mu'}{\Rightarrow}$ *is a subderivation of* $t\|(s'.X) \overset{\sigma'}{\Rightarrow}$ *from* s'*, then* $s \overset{r}{\Rightarrow} s' \overset{\mu'}{\Rightarrow}$ *is a subderivation of* d *from* s*;*
3. *if* d *is of the form* $t\|(s.X) \overset{r}{\Rightarrow} t'\|(s.X) \overset{\sigma'}{\Rightarrow}$ *(with* $t \overset{r}{\Rightarrow} t'$ *and* $r \in \Re$ *), then every subderivation of* $t'\|(s.X) \overset{\sigma'}{\Rightarrow}$ *from* s *is also a subderivation of* d *from* s*.*

Moreover, we say that d' *is a subderivation of* $\overline{t} \overset{\lambda}{\Rightarrow} t\|(s.X) \overset{\sigma}{\Rightarrow}$*.*

Given a rule sequence σ in $\Re$, and a subsequence σ' of σ, $\sigma \setminus \sigma'$ denotes the rule sequence obtained by removing from σ all and only the occurrences of rules in σ'.

STEP 1 We prove decidability of Problem **1** by a reduction to a similar problem restricted to a parallel *MBRS* (that is decidable by Proposition 2). The main idea is to mimic finite derivations in M of the form $p \overset{\sigma}{\Rightarrow} t$ (preserving p, the finite acceptance of σ in M, and the final term t if $t \in T_{PAR}$) starting from terms in T_{PAR} by using only PAR rules belonging to an extension, denoted by M_{PAR} (and with support $\Re_{PAR}$), of the parallel *MBRS* M_P. In order to illustrate this, let us denote by $N_{SEQ}(\sigma)$ the number of occurrences in σ of SEQ rules of the form $X \overset{a}{\rightarrow} Z.Y$. We proceed by induction on $N_{SEQ}(\sigma)$. If $N_{SEQ}(\sigma) = 0$, since $p \in T_{PAR}$, we deduce that $p \overset{\sigma}{\Rightarrow} t$ is also a derivation in M_P (and so in M_{PAR}, since M_{PAR} is an extension of M_P). Now, let us assume that $N_{SEQ}(\sigma) > 0$. In this case $p \overset{\sigma}{\Rightarrow} t$ can be rewritten in the form $p \overset{\lambda}{\Rightarrow} \overline{p}\|X \overset{r}{\Rightarrow} \overline{p}\|(Z.Y) \overset{\nu}{\Rightarrow} t$ where $r = X \overset{a}{\rightarrow} Z.Y$, λ contains only occurrences of PAR rules in $\Re$, $\overline{p} \in T_{PAR}$ and $X, Y, Z \in Var$. Let $Z \overset{\rho}{\Rightarrow} t_1$ be a subderivation of $\overline{p}\|(Z.Y) \overset{\nu}{\Rightarrow} t$ from Z. By the definition of subderivation only one of the following three cases may occur:

[4] In the following, locutions of the kind 'the derivation $t \overset{\sigma}{\Rightarrow}$' mean that (there is a derivation of this form) and we are considering a specific derivation of the form $t \overset{\sigma}{\Rightarrow}$, and $t \overset{\sigma}{\Rightarrow}$ is used as a reference to this derivation.

A $t_1 \neq \varepsilon$, $\overline{p} \overset{\nu\setminus\rho}{\Rightarrow} t_2$, and $t = t_2 \| (t_1.Y)$.

B $t_1 = \varepsilon$ and $p \overset{\sigma}{\Rightarrow} t$ is of the form $p \overset{\lambda}{\Rightarrow} \overline{p} \| X \overset{r}{\Rightarrow} \overline{p} \| (Z.Y) \overset{\nu_1}{\Rightarrow} t_2 \| Y \overset{\nu_2}{\Rightarrow} t$, where ρ is a subsequence of ν_1 and $\overline{p} \overset{\nu_1\setminus\rho}{\Rightarrow} t_2$.

C $t_1 = W \in Var$, and $p \overset{\sigma}{\Rightarrow}$ can be written as

$$p \overset{\lambda}{\Rightarrow} \overline{p} \| X \overset{r}{\Rightarrow} \overline{p} \| (Z.Y) \overset{\nu_1}{\Rightarrow} t_2 \| (W.Y) \overset{r'}{\Rightarrow} t_2 \| W' \overset{\nu_2}{\Rightarrow} t \tag{1}$$

where $r' = W.Y \overset{b}{\rightarrow} W'$, ρ is a subsequence of ν_1 and $\overline{p} \overset{\nu_1\setminus\rho}{\Rightarrow} t_2$.

Cases **A**, **B** and **C** can be dealt in similar way, so that we examine only case **C**. Let us consider equation (1). By anticipating the application of the rules in $\rho r'$ before the application of the rules in $\nu_1 \setminus \rho$ we obtain the following derivation that has the same finite acceptance as (1): $p \overset{\lambda}{\Rightarrow} \overline{p} \| X \overset{r}{\Rightarrow} \overline{p} \| (Z.Y) \overset{\rho}{\Rightarrow} \overline{p} \| (W.Y) \overset{r'}{\Rightarrow} \overline{p} \| W' \overset{\gamma}{\Rightarrow} t$, where $\gamma = (\nu_1 \setminus \rho)\nu_2$. Since $Z \overset{\rho}{\Rightarrow} W$ with $Z, W \in Var$ and $N_{SEQ}(\rho) < N_{SEQ}(\sigma)$, by the induction hypothesis there will be a derivation in M_{PAR} having the form $Z \overset{\pi}{\Rightarrow}_{\Re_{PAR}} W$ with $\Upsilon^f_{M_{PAR}}(\pi) = \Upsilon^f_M(\rho)$. By Proposition 2 for each $K' \in P_n$ it is decidable whether there exists in M_{PAR} a finite derivation starting from variable Z and leading to variable W, having finite acceptance K' (in M_{PAR}). Then, the idea is to collapse the finite derivation $d = X \overset{r}{\Rightarrow} Z.Y \overset{\rho}{\Rightarrow} W.Y \overset{r'}{\Rightarrow} W'$ into a single PAR rule of the form $r'' = X \overset{K'}{\rightarrow} W'$ such that $K' = \Upsilon^f_M(rr') \cup \Upsilon^f_{M_{PAR}}(\pi) = \Upsilon^f_M(rr'\rho)$ and $\Upsilon^f_{M_{PAR}}(r'') = K'$. So, rule r'' keeps track of the meaningful information of the derivation d, i.e., the starting term $X \in Var$, the final term $W' \in Var$, and the finite acceptance of $rr'\rho$ in M. Since the set of rules of the form $X \overset{K'}{\rightarrow} W'$ with $X, W' \in Var$ and $K' \in P_n$ is finite, M_{PAR} can be built effectively. After all, we have that $p \overset{\lambda r''}{\Rightarrow}_{\Re_{PAR}} \overline{p} \| W'$ and $\overline{p} \| W' \overset{\gamma}{\Rightarrow}_{\Re} t$ such that $\overline{p} \| W' \in T_{PAR}$, $\Upsilon^f_{M_{PAR}}(\lambda r'') = \Upsilon^f_M(\lambda r r' \rho)$ and $N_{SEQ}(\gamma) < N_{SEQ}(\sigma)$. Applying again the induction hypothesis we deduce that there exists a finite derivation in M_{PAR} of the form $p \overset{\xi}{\Rightarrow}_{\Re_{PAR}} p'$ such that $\Upsilon^f_{M_{PAR}}(\xi) = \Upsilon^f_M(\sigma)$, and $p' = t$ if $t \in T_{PAR}$. The fresh variable Z_F is used to manage case **A**, where the subderivation $Z \overset{\rho}{\Rightarrow} t_1$ does not influence the applicability of rules in $\nu \setminus \rho$ (i.e., $\overline{p} \overset{\nu\setminus\rho}{\Rightarrow} t_2$). In this case, in order to keep track of the derivation $X \overset{r}{\Rightarrow} Z.Y \overset{\rho}{\Rightarrow} t_1.Y$, it suffices to preserve the starting term X and the finite acceptance of $r\rho$. Therefore, we introduce a new rule of the form $r'' = X \overset{K'}{\rightarrow} Z_F$ such that $K' = \Upsilon^f_M(r\rho)$ and $\Upsilon^f_{M_{PAR}}(r'') = K'$. M_{PAR} is formally defined as follows.

Definition 5. *The* MBRS $M_{PAR} = \langle \Re_{PAR}, \langle \Re_{PAR,1}, \ldots, \Re_{PAR,n} \rangle \rangle$ *is the least parallel* MBRS*, over* $Var \cup \{Z_F\}$ *and the alphabet* $\Sigma \cup P_n$*, such that:*

1. $\Re_{PAR} \supseteq \Re_P$ *and* $\Re_{PAR,i} \supseteq \Re_{P,i}$ *for all* $i = 1, \ldots, n$.
2. *Let* $r = X \overset{a}{\rightarrow} Z.Y \in \Re$, $Z \overset{\sigma}{\Rightarrow}_{\Re_{PAR}} p$ *for some term* p *(resp.,* $Z \overset{\sigma}{\Rightarrow}_{\Re_{PAR}} \varepsilon$*), and* $K' = \Upsilon^f_M(r) \cup \Upsilon^f_{M_{PAR}}(\sigma)$*. Then* $r' = X \overset{K'}{\rightarrow} Z_F \in \Re_{PAR}$ *(resp.,* $r' = X \overset{K'}{\rightarrow} Y \in \Re_{PAR}$*) and* $\Upsilon^f_{M_{PAR}}(r') = K'$.

3. *Let* $r = X \xrightarrow{a} Z.Y \in \Re$, $r' = W.Y \xrightarrow{b} W' \in \Re$, $Z \overset{\sigma}{\Rightarrow}_{\Re_{PAR}} W$, *and* $K' = \Upsilon^f_M(rr') \cup \Upsilon^f_{M_{PAR}}(\sigma)$. *Then* $r'' = X \xrightarrow{K'} W' \in \Re_{PAR}$ *and* $\Upsilon^f_{M_{PAR}}(r'') = K'$.

Lemma 1. *The parallel* MBRS M_{PAR} *can be effectively constructed.*

Proof. Figure 1 reports the procedure BUILD-PARALLEL-MBRS(M) which builds M_{PAR}. The algorithm uses the routine $UPDATE(r', K')$ defined as:
$\Re_{PAR} := \Re_{PAR} \cup \{r'\}$;
for each $i \in K'$ **do** $\Re_{PAR,i} := \Re_{PAR,i} \cup \{r'\}$;
Notice that by Proposition 2, the conditions in each of the **if** statements in lines 7, 9 and 13 are decidable, therefore, the procedure is effective. Moreover, since the set of rules of the form $X \xrightarrow{K'} Y$ with $X \in Var$, $Y \in Var \cup \{Z_F\}$ and $K' \in P_n$ is finite, termination is guaranteed.

Algorithm BUILD–PARALLEL–MBRS(M)

1 $\Re_{PAR} := \Re_P$;
2 **for** $i = 1, \ldots, n$ **do** $\Re_{PAR,i} := \Re_{P,i}$;
3 **repeat**
4 *flag:=false;*
5 **for each** $r = X \xrightarrow{a} Z.Y \in \Re$ *and* $K_1 \in P_n$ **do**
6 *Set* $K' = K_1 \cup \Upsilon^f_M(r)$;
7 **if** $Z \overset{\sigma}{\Rightarrow}_{\Re_{PAR}} p$ *for some* p *such that* $\Upsilon^f_{M_{PAR}}(\sigma) = K_1$ **then**
8 **if** $r' = X \xrightarrow{K'} Z_F \notin \Re_{PAR}$ **then** $UPDATE(r', K')$; *flag:=true;*
9 **if** $Z \overset{\sigma}{\Rightarrow}_{\Re_{PAR}} \varepsilon$ *such that* $\Upsilon^f_{M_{PAR}}(\sigma) = K_1$ **then**
10 **if** $r' = X \xrightarrow{K'} Y \notin \Re_{PAR}$ **then** $UPDATE(r', K')$; *flag:=true;*
11 **for each** $r' = W.Y \xrightarrow{b} W' \in \Re$ **do**
12 *Set* $K' = K_1 \cup \Upsilon^f_M(rr')$;
13 **if** $Z \overset{\sigma}{\Rightarrow}_{\Re_{PAR}} W$ *such that* $\Upsilon^f_{M_{PAR}}(\sigma) = K_1$ **then**
14 **if** $r'' = X \xrightarrow{K'} W' \notin \Re_{PAR}$ **then** $UPDATE(r'', K')$; *flag:=true;*
15 **until** *flag* = *false*

Fig. 1. Algorithm to build the parallel *MBRS* M_{PAR}.

The following two lemmata (whose proof is simple) establish the correctness of our construction.

Lemma 2. *Let* $p \overset{\sigma}{\Rightarrow}_{\Re} t \| p'$ *with* $p, p' \in T_{PAR}$. *Then, there exists* $s \in T_{PAR}$ *such that* $p \overset{\rho}{\Rightarrow}_{\Re_{PAR}} s \| p'$, $\Upsilon^f_M(\sigma) = \Upsilon^f_{M_{PAR}}(\rho)$, $s = \varepsilon$ *if* $t = \varepsilon$, *and* $|\rho| > 0$ *if* $|\sigma| > 0$.

Lemma 3. *Let* $p \overset{\sigma}{\Rightarrow}_{\Re_{PAR}} p' \| p''$ *such that* $p, p', p'' \in T_{PAR}$, p' *does not contain occurrences of* Z_F, *and* p'' *does not contain occurrences of variables in* Var. *Then, there exists* $t \in T$ *such that* $p \overset{\rho}{\Rightarrow}_{\Re} p' \| t$, $\Upsilon^f_M(\rho) = \Upsilon^f_{M_{PAR}}(\sigma)$, $t = \varepsilon$ *if* $p'' = \varepsilon$, *and* $|\rho| > 0$ *if* $|\sigma| > 0$.

These two results, together with Proposition 2, allow us to conclude that Problem **1** is decidable.

STEP 2 Let us go back to the Fairness Problem. We define a class of derivations, in symbols $\Pi(K, K_\omega)$, that is the set of derivations d in $\Re$ such that there is *not* a subderivation of d that is a (K, K_ω)-accepting infinite derivation in M. Now, we show that we can limit ourselves to consider only this class of derivations. Let d be a (K, K_ω)-accepting infinite derivation in M from a variable X. If d does *not* belong to $\Pi(K, K_\omega)$, then it can be written in the form $X \overset{\rho}{\Rightarrow} t \| W \overset{r}{\Rightarrow} t \| (Z.Y) \overset{\nu}{\Rightarrow}$, with $Z \in Var$ and $r = W \overset{a}{\rightarrow} Z.Y$, and such that there exists a subderivation of $t \| (Z.Y) \overset{\nu}{\Rightarrow}$ from Z that is a (K, K_ω)-accepting infinite derivation in M. Following this argument we can prove that there exist $m \in \mathbb{N} \setminus \{0\} \cup \{\infty\}$, a sequence of variables $(X_h)_{h=0}^{h=m}$ with $X_0 = X$, and a sequence of SEQ rules $(r_h)_{h=1}^{h=m}$ such that one of the following two conditions is satisfied:

1. m is finite, for each $h = 0, \ldots, m-1$ we have that $X_h \overset{\rho_h}{\Rightarrow} t_h \| Y_{h+1}$, $r_{h+1} = Y_{h+1} \overset{a_{h+1}}{\rightarrow} X_{h+1}.Z_{h+1}$, $\Upsilon^f_M(\rho_h r_{h+1}) \subseteq K$, and there exists a (K, K_ω)-accepting infinite derivation in M from X_m *belonging* to $\Pi(K, K_\omega)$.
2. (for $K = K_\omega$) m is infinite, and for all $h \in \mathbb{N}$ we have that $X_h \overset{\rho_h}{\Rightarrow} t_h \| Y_{h+1}$, $r_{h+1} = Y_{h+1} \overset{a_{h+1}}{\rightarrow} X_{h+1}.Z_{h+1}$, and $\Upsilon^f_M(\rho_0 r_1 \rho_1 r_2 \ldots) = \Upsilon^\infty_M(\rho_0 r_1 \rho_1 r_2 \ldots) = K_\omega$.

For each h let us consider the derivation $X_h \overset{\rho_h}{\Rightarrow} t_h \| Y_{h+1}$. By Lemma 2 there exists a finite derivation in M_{PAR} of the form $X_h \overset{\lambda_h}{\Rightarrow}_{\Re_{PAR}} p_h \| Y_{h+1}$ such that $\Upsilon^f_{M_{PAR}}(\lambda_h) = \Upsilon^f_M(\rho_h)$ and $p_h \in T_{PAR}$. By Proposition 2 for each $K' \in P_n$ it is decidable whether variable Y_{h+1} is partially reachable in M_{PAR} from X_h through a derivation having finite acceptance K'. The idea is to introduce a SEQ rule of the form $X_h \overset{K'}{\rightarrow} Y_{h+1}$ where $K' = \Upsilon^f_{M_{PAR}}(\lambda_h)$, and whose finite acceptance is K'. Let us denote by M_{SEQ} the sequential *MBRS* (with n accepting components) containing these new rules (whose number is finite) and all the SEQ rules of M having the form $X \overset{a}{\rightarrow} Z.Y$, and whose accepting components agree with the labels of the new rules. Then, case 2 above amounts to check the existence of a (K, K_ω)-accepting infinite derivation in M_{SEQ} from variable X. By Proposition 4 this is decidable. Case 1 amounts to check the existence of a variable $Y \in Var$ such that Y is s-reachable from X in M_{SEQ} through a derivation with finite acceptance (in M_{SEQ}) $K' \subseteq K$ (by Proposition 4 this is decidable), and there exists a (K, K_ω)-accepting infinite derivation in M from Y belonging to $\Pi(K, K_\omega)$. M_{SEQ} is formally defined as follows.

Definition 6. *By* $M_{SEQ} = \langle \Re_{SEQ}, \langle \Re_{SEQ,1}, \ldots, \Re_{SEQ,n} \rangle \rangle$ *we denote the* sequential MBRS *over* Var *and the alphabet* $\Sigma \cup P_n$ *defined as follows:*

– $\Re_{SEQ} = \{X \overset{a}{\rightarrow} Z.Y \in \Re\} \cup$
$\{X \overset{K'}{\rightarrow} Y \mid X, Y \in Var,\ X \overset{\sigma}{\Rightarrow}_{\Re_{PAR}} p \| Y$ *for some* $p \in T_{PAR}$, $|\sigma| > 0$, *and* $\Upsilon^f_{M_{PAR}}(\sigma) = K'\}$.

– $\Re_{SEQ,i} = \{X \overset{a}{\rightarrow} Z.Y \in \Re_i\} \cup \{X \overset{K'}{\rightarrow} Y \in \Re_{SEQ} \mid i \in K'\}$ *for all* $i = 1, \ldots, n$.

By Proposition 2 we obtain the following result

Lemma 4. *M_{SEQ} can be built effectively.*

Thus, we obtain a first reduction of the Fairness Problem.

Lemma 5. *Given $X \in Var$, there exists a (K, K_ω)-accepting infinite derivation in M from X if, and only if, one of the following conditions is satisfied:*

1. *There exists a variable $Y \in Var$ s-reachable from X in $\Re_{SEQ}$ through a $(K', \emptyset)$-accepting derivation in M_{SEQ} with $K' \subseteq K$, and there exists a (K, K_ω)-accepting infinite derivation in M from Y belonging to $\Pi(K, K_\omega)$.*
2. *(Only when $K = K_\omega$) There exists a (K, K_ω)-accepting infinite derivation in M_{SEQ} from X.*

Therefore, it remains to manage the class $\Pi(K, K_\omega)$. We proceed by induction on $|K| + |K_\omega|$. Let $p \overset{\sigma}{\Rightarrow}$ be a (K, K_ω)-accepting infinite derivation in M from $p \in T_{PAR}$ belonging to $\Pi(K, K_\omega)$. If σ contains only occurrences of PAR rules, then $p \overset{\sigma}{\Rightarrow}$ is also a (K, K_ω)-accepting infinite derivation in M_{PAR}. Otherwise, it can be rewritten in the form $p \overset{\lambda}{\Rightarrow} \overline{p} \| W \overset{r}{\Rightarrow} \overline{p} \| (Z.Y) \overset{\nu}{\Rightarrow}$ where $r = W \overset{a}{\rightarrow} Z.Y$, λ contains only occurrences of PAR rules in $\Re$, $\overline{p} \in T_{PAR}$ and $W, Y, Z \in Var$. Let $Z \overset{\rho}{\Rightarrow}$ be a subderivation of $\overline{p} \| (Z.Y) \overset{\nu}{\Rightarrow}$ from Z. If $Z \overset{\rho}{\Rightarrow}$ is finite, as shown in *Step 1*, we can keep track of the finite derivation $W \overset{r}{\Rightarrow}$ Z.Y $\overset{\rho}{\Rightarrow}$ (preserving acceptance properties) by using a PAR rule belonging to M_{PAR}. If $|K| + |K_\omega| = 0$ (i.e., $K = K_\omega = \emptyset$), since $p \overset{\sigma}{\Rightarrow}$ belongs to $\Pi(K, K_\omega)$ (and ρ is a subsequence of σ), then $Z \overset{\rho}{\Rightarrow}$ can be only finite. Therefore, all the subderivations of $p \overset{\sigma}{\Rightarrow}$ are finite. Then, by Step 1 we deduce that there must exist a $(\emptyset, \emptyset)$-accepting infinite derivation in M_{PAR} from p. Since $Var \subseteq T_{PAR}$, by Lemma 5 we obtain the following decidable (by Propositions 3–4) characterization for the existence of a $(\emptyset, \emptyset)$-accepting infinite derivation in M from a variable X:

- (when $K = K_\omega = \emptyset$) Either (1) there exists a $(\emptyset, \emptyset)$-accepting infinite derivation in M_{SEQ} from X, or (2) there exists a variable Y s-reachable from X in M_{SEQ} through a derivation having finite acceptance (in M_{SEQ}) $K = \emptyset$, and there exists a $(\emptyset, \emptyset)$-accepting infinite derivation in M_{PAR} from Y.

Now, let us assume that $|K| + |K_\omega| > 0$ and $Z \overset{\rho}{\Rightarrow}$ is infinite. Since $\overline{p} \| (Z.Y) \overset{\nu}{\Rightarrow}$ is also in $\Pi(K, K_\omega)$, by definition of subderivation we deduce that there is a derivation belonging to $\Pi(K, K_\omega)$ having the form $\overline{p} \overset{\nu \setminus \rho}{\Rightarrow}$. Since $p \overset{\sigma}{\Rightarrow}$ belongs to $\Pi(K, K_\omega)$, it follows that $\Upsilon^f_M(\rho) = \overline{K} \subseteq K$, $\Upsilon^\infty_M(\rho) = \overline{K}_\omega \subseteq K_\omega$, and $|\overline{K}| + |\overline{K}_\omega| < |K| + |K_\omega|$. By our assumptions (induction hypothesis) it is decidable whether there exists a $(\overline{K}, \overline{K}_\omega)$-accepting infinite derivation in M from variable Z. Then, we keep track of the infinite derivation $W \overset{r}{\Rightarrow}_\Re Z.Y \overset{\rho}{\Rightarrow}_\Re$ by adding a PAR rule of the form $r' = W \overset{K_1, \overline{K}_\omega}{\rightarrow} Z_F$ with $K_1 = \overline{K} \cup \Upsilon^f_M(r) \subseteq K$. So, the label of r' keeps track of the finite and infinite acceptance of $r\rho$ in M. Now, we can apply recursively the same argument to the derivation $\overline{p} \| Z_F \overset{\nu \setminus \rho}{\Rightarrow}$ in $\Re$ from $\overline{p} \| Z_F \in T_{PAR}$, which belongs to $\Pi(K, K_\omega)$ and whose finite (resp., infinite) acceptance in M is contained in K (resp., K_ω). In other words, all the subderivations of

$p \overset{\sigma}{\Rightarrow}$ are abstracted away by PAR rules non belonging to $\Re$, according to the intuitions given above. Formally, we define two extensions of M_{PAR} (with the same support) that will contain these new PAR rules $r' = W \overset{K_1,\overline{K}_\omega}{\rightarrow} Z_F$. The accepting components of the first (resp., the second) extension agree with the first component K_1 (resp., the second component $\overline{K}_\omega$) of the label of r' (that keep track of the finite acceptance – resp., the infinite acceptance – of the simulated infinite rule sequences in M).

Definition 7. *By* $M_{PAR}^{K,K_\omega} = \langle \Re_{PAR}^{K,K_\omega}, \langle \Re_{PAR,1}^{K,K_\omega}, \ldots, \Re_{PAR,n}^{K,K_\omega} \rangle\rangle$ *and* $M_{PAR,\infty}^{K,K_\omega} = \langle \Re_{PAR}^{K,K_\omega}, \langle \Re_{PAR,\infty,1}^{K,K_\omega}, \ldots, \Re_{PAR,\infty,n}^{K,K_\omega} \rangle\rangle$ *we denote the parallel* MBRS*s over* $Var \cup \{Z_F\}$ *and the alphabet* $\Sigma \cup P_n \cup P_n \times P_n$ *(with the same support), defined as follows:*

– $\Re_{PAR}^{K,K_\omega} = \Re_{PAR} \cup$
$$\{X \overset{\overline{K},\overline{K}_\omega}{\rightarrow} Z_F \mid \overline{K} \subseteq K, \overline{K}_\omega \subseteq K_\omega, \text{ there exist } r = X \overset{a}{\rightarrow} Z.Y \in \Re \text{ and an infinite derivation } Z \overset{\sigma}{\Rightarrow}_{\Re} \text{ such that } |\Upsilon_M^f(\sigma)| + |\Upsilon_M^\infty(\sigma)| < |K| + |K_\omega| \text{ and } \Upsilon_M^f(\sigma) \cup \Upsilon_M^f(r) = \overline{K} \text{ and } \Upsilon_M^\infty(\sigma) = \overline{K}_\omega\}.$$
– $\Re_{PAR,i}^{K,K_\omega} = \Re_{PAR,i} \cup \{X \overset{\overline{K},\overline{K}_\omega}{\rightarrow} Z_F \in \Re_{PAR}^{K,K_\omega} \mid i \in \overline{K}\}$ *for all* $i = 1, \ldots, n$.
– $\Re_{PAR,i,\infty}^{K,K_\omega} = \{X \overset{\overline{K},\overline{K}_\omega}{\rightarrow} Z_F \in \Re_{PAR}^{K,K_\omega} \mid i \in \overline{K}_\omega\}$ *for all* $i = 1, \ldots, n$.

By the induction hypothesis on the decidability of the Fairness Problem for sets $\overline{K}, \overline{K}_\omega \in P_n$ such that $\overline{K} \subseteq K$, $\overline{K}_\omega \subseteq K_\omega$ and $|\overline{K}| + |\overline{K}_\omega| < |K| + |K_\omega|$, we have

Lemma 6. M_{PAR}^{K,K_ω} *and* $M_{PAR,\infty}^{K,K_\omega}$ *can be built effectively.*

The following two Lemmata establish the correctness of our construction.

Lemma 7. *Let* $p \overset{\sigma}{\Rightarrow}$ *be a* $(\overline{K}, \overline{K}_\omega)$*-accepting derivation in* M *from* $p \in T_{PAR}$ *belonging to* $\Pi(K, K_\omega)$*, where* $\overline{K} \subseteq K$ *and* $\overline{K}_\omega \subseteq K_\omega$*. Then, there exists in* $\Re_{PAR}^{K,K_\omega}$ *a derivation of the form* $p \overset{\rho}{\Rightarrow}$ *such that* $\Upsilon^f_{M_{PAR}^{K,K_\omega}}(\rho) = \overline{K}$*,* $\Upsilon^\infty_{M_{PAR}^{K,K_\omega}}(\rho) \cup \Upsilon^f_{M_{PAR,\infty}^{K,K_\omega}}(\rho) = \overline{K}_\omega$*. Moreover, if* σ *is infinite, then either* ρ *is infinite or contains some occurrence of rule in* $\Re_{PAR}^{K,K_\omega} \setminus \Re_{PAR}$*.*

Lemma 8. *Let* $p \overset{\sigma}{\Rightarrow}_{\Re_{PAR}^{K,K_\omega}}$ *such that* $p \in T_{PAR}$*, and* σ *is either infinite or contains some occurrence of rule in* $\Re_{PAR}^{K,K_\omega} \setminus \Re_{PAR}$*. Then, there exists in* $\Re$ *an infinite derivation of the form* $p \overset{\delta}{\Rightarrow}$ *such that* $\Upsilon^f_M(\delta) = \Upsilon^f_{M_{PAR}^{K,K_\omega}}(\sigma)$ *and* $\Upsilon^\infty_M(\delta) = \Upsilon^\infty_{M_{PAR}^{K,K_\omega}}(\sigma) \cup \Upsilon^f_{M_{PAR,\infty}^{K,K_\omega}}(\sigma)$*.*

Finally, we can prove the desired result

Theorem 2. *The Fairness Problem is decidable for* MBRS*s in normal form.*

Proof. We start constructing M_{PAR} and M_{SEQ} (they do not depend on K and K_ω). Then, we accumulate information about the existence of $(\overline{K}, \overline{K}_\omega)$-accepting infinite derivations in M from variables in Var, where $|\overline{K}| + |\overline{K}_\omega| \leq |K| + |K_\omega|$ and $\overline{K} \subseteq K$ and $\overline{K}_\omega \subseteq K_\omega$, proceeding for crescent values of $|\overline{K}|+|\overline{K}_\omega|$. We have seen that this is decidable for $|\overline{K}| + |\overline{K}_\omega| = 0$. We keep track of this information by adding new PAR rules according to Definition 7. When $|\overline{K}| + |\overline{K}_\omega| > 0$ (assuming without loss of generality that $\overline{K} = K$ and $\overline{K}_\omega = K_\omega$), by Lemmata 5, 7, and 8 the problem for a variable $X \in Var$ is reduced to check that one of the following two conditions (that are decidable by Propositions 3–4) holds:

– There exists a variable $Y \in Var$ s-reachable from X in $\Re_{SEQ}$ through a $(K', \emptyset)$-accepting derivation in M_{SEQ} with $K' \subseteq K$, and there exists a derivation $Y \overset{\rho}{\Rightarrow}_{\Re^{K,K_\omega}_{PAR}}$ such that $\Upsilon^f_{M^{K,K_\omega}_{PAR}}(\rho) = K$ and $\Upsilon^\infty_{M^{K,K_\omega}_{PAR}}(\rho) \cup \Upsilon^f_{M^{K,K_\omega}_{PAR,\infty}}(\rho) = K_\omega$. Moreover, ρ is either infinite or contains some occurrence of rule in $\Re^{K,K_\omega}_{PAR} \setminus \Re_{PAR}$.
– (only when $K = K_\omega$). There exists a (K, K_ω)-accepting infinite derivation in M_{SEQ} from X.

5 Decidability of the Fairness Problem for Unrestricted *MBRS*s

In this section we extend the decidability result stated in the previous Section to the whole class of *MBRS*s, showing that the Fairness Problem for unrestricted *MBRS*s is reducible to the Fairness Problem for *MBRS*s in normal form. We use a construction very close to that used in [12, 13] to solve reachability for *PRSs*. We recall that we can assume that the input term in the Fairness Problem is a process variable. Let M be a *MBRS* over Var and Σ, and with n accepting components. Now, we describe a procedure that transforms M into a new *MBRS* M' with the same number of accepting components. Moreover, this procedure has as input also a finite set of rules $\Re_{AUX}$, and transforms it in $\Re'_{AUX}$. If M is not in normal form, then there exists some rule r in M (that we call *bad rule* [12]) that is neither a PAR rule nor a SEQ rule. There are five types of bad rules[5]:

1. $r = u \xrightarrow{a} u_1 \| u_2$. Let Z_1, Z_2, W be fresh variables. We get M' replacing the bad rule r with the rules $r' = u \rightarrow W$, $r_3 = W \rightarrow Z_1 \| Z_2$, $r_1 = Z_1 \rightarrow u_1$, $r_2 = Z_2 \rightarrow u_2$ such that $\Upsilon^f_{M'}(r') = \Upsilon^f_M(r)$, $\Upsilon^f_{M'}(r_1) = \Upsilon^f_{M'}(r_2) = \Upsilon^f_{M'}(r_3) = \emptyset$ [6]. If $r \in \Re_{AUX}$, then $\Re'_{AUX} = (\Re_{AUX} \setminus \{r\}) \cup \{r', r_1, r_2, r_3\}$, otherwise, $\Re'_{AUX} = \Re_{AUX}$.
2. $r = u_1 \| (u_2.u_3) \xrightarrow{a} u$. Let Z_1, Z_2 be fresh variables. We get M' replacing the bad rule r with the rules $r_1 = u_1 \rightarrow Z_1$, $r_2 = u_2.u_3 \rightarrow Z_2$, $r' = Z_1 \| Z_2 \rightarrow u$ such that $\Upsilon^f_{M'}(r') = \Upsilon^f_M(r)$, $\Upsilon^f_{M'}(r_1) = \Upsilon^f_{M'}(r_2) = \emptyset$. If $r \in \Re_{AUX}$, then $\Re'_{AUX} = (\Re_{AUX} \setminus \{r\}) \cup \{r', r_1, r_2\}$, otherwise, $\Re'_{AUX} = \Re_{AUX}$.

[5] We assume that sequential composition is left-associative. So, when we write $t_1.t_2$, then t_2 is either a single variable or a parallel composition of process terms.

[6] Note that we have not specified the label of the new rules, since it is not relevant.

3. $r = u \xrightarrow{a} u_1.u_2$ (resp., $r = u_1.u_2 \xrightarrow{a} u$) where u_2 is not a single variable. Let Z be a fresh variable. We get M' and $\Re'_{AUX}$ in two steps. First, we substitute Z for u_2 in (left-hand and right-hand sides of) all the rules of M and $\Re_{AUX}$. Then, we add the rules $r_1 = Z \rightarrow u_2$ and $r_2 = u_2 \rightarrow Z$ such that $\Upsilon^f_{M'}(r_1) = \Upsilon^f_{M'}(r_2) = \emptyset$.
4. $r = u_1 \xrightarrow{a} u_2.X$. Let Z, W be fresh variables. We get M' replacing the bad rule r with the rules $r' = u_1 \rightarrow W$, $r_1 = W \rightarrow Z.X$, $r_2 = Z \rightarrow u_2$ such that $\Upsilon^f_{M'}(r') = \Upsilon^f_M(r)$ and $\Upsilon^f_{M'}(r_1) = \Upsilon^f_{M'}(r_2) = \emptyset$. If $r \in \Re_{AUX}$, then $\Re'_{AUX} = (\Re_{AUX} \setminus \{r\}) \cup \{r', r_1, r_2\}$, otherwise, $\Re'_{AUX} = \Re_{AUX}$.
5. $r = u_1.X \xrightarrow{a} u_2$ where u_1 is not a single variable. Let Z be a fresh variable. We get M' replacing the bad rule r with the rules $r_1 = u_1 \rightarrow Z$, $r' = Z.X \rightarrow u_2$, such that $\Upsilon^f_{M'}(r') = \Upsilon^f_M(r)$ and $\Upsilon^f_{M'}(r_1) = \emptyset$. If $r \in \Re_{AUX}$, then $\Re'_{AUX} = (\Re_{AUX} \setminus \{r\}) \cup \{r', r_1\}$, otherwise, $\Re'_{AUX} = \Re_{AUX}$.

After a finite number of applications of this procedure, starting from $\Re_{AUX} = \emptyset$, we obtain a *MBRS* M' in normal form and a finite set of rules $\Re'_{AUX}$. Let $M' = \langle \Re', \langle \Re'_1, \ldots, \Re'_n \rangle \rangle$. Now, let us consider the *MBRS* in normal form with $n+1$ accepting components given by $M_F = \langle \Re', \langle \Re'_1, \ldots, \Re'_n, \Re' \setminus \Re'_{AUX} \rangle \rangle$. We can prove that, given a variable $X \in Var$ and two sets $K, K_\omega \in P_n$, there exists a (K, K_ω)-accepting infinite derivation in M from X if, and only if, there exists a $(K \cup \{n+1\}, K_\omega \cup \{n+1\})$-accepting infinite derivation in M_F from X.

6 Complexity Issues

We conclude with some considerations about the complexity of the considered problem. Model–checking parallel *PRSs* (that are equivalent to Petri nets) w.r.t. the considered *ALTL* fragment, interpreted on infinite runs, is *EXPSPACE*-complete (also for a fixed formula) [11]. *ALTL* model–checking for sequential *PRSs* (that are equivalent to Pushdown processes) is less hard, since it is *EXPTIME*-complete [2]. Therefore, model–checking the whole class of *PRSs* w.r.t. the considered *ALTL* fragment (restricted to infinite runs) is at least *EXPSPACE*-hard. We have reduced this problem (in polynomial time) to the Fairness Problem (see Theorem 1). Moreover, as seen in Section 5, we can limit ourselves (through a polynomial-time reduction) to consider only *MBRS*s in normal form. The algorithm presented in Section 4 to resolve the Fairness Problem for *MBRS*s in normal form is an exponential reduction (in the number n of accepting components) to the *ALTL* model–checking problem for Petri nets and Pushdown processes: we have to resolve an exponential number in n of instances of decision problems about acceptance properties of derivations of parallel and sequential *MBRS*s, whose size is exponential in n [7]. These last problems (see Propositions 2–4) are polynomial-time reducible to the *ALTL* model–checking problem for Petri nets and Pushdown processes. It was shown [8] that for Petri nets, and for

[7] Note that the number of new rules added in order to built M_{PAR}, M_{SEQ}, M^{K,K_ω}_{PAR}, and $M^{K,K_\omega}_{PAR,\infty}$ is exponential in n and polynomial in $|Var|$.

a fixed *ALTL* formula, model checking has the same complexity as reachability (that is *EXPSPACE*-hard, but the best known upper bound is not primitive recursive). Therefore, for n fixed (i.e., for a fixed formula of our *ALTL* fragment) the upper bound given by our algorithm is the same as reachability for Petri nets.

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