Strong termination for gap-order constraint abstractions of counter systems

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Abstract. We address termination analysis for the class of *gap-order constraint systems* (GCS), an (infinitely-branching) abstract model of counter machines recently introduced in [7], in which constraints (over $\mathbb{Z}$) between the variables of the source state and the target state of a transition are *gap-order constraints* (GC) [17]. GCS extend monotonicity constraint systems [4], integral relation automata [8], and constraint automata in [11]. Since GCS are infinitely-branching, termination does not imply *strong termination*, i.e. the existence of an upper bound on the lengths of the runs from a given state. We show the following: (1) checking strong termination for GCS is decidable and PSPACE-complete, and (2) for each control location of the given GCS, one can built a GC representation of the set of variable valuations from which strong termination does *not* hold.

1 Introduction

Abstractions of Counter Systems. One standard approach in formal analysis is the abstraction based one: the analysis is performed on an *abstraction* of the given system, specified in some weak (non-complete) computational formalism for which checking the properties of interest is decidable. The relation between the abstraction and the concrete system is usually specified as a semantic over-approximation. This ensures that the approach is conservative, by giving a decision procedure that (for correct systems) is sound but in general incomplete. With regard to the class of counter systems, a widely investigated complete computational model, interesting abstractions have been studied, for which meaningful classes of verification problems have been shown to be decidable. Many of these abstractions are in fact restrictions: examples include Petri nets [15], reversal-bounded counter machines [13], and flat counter systems [9]. Genuine abstractions are obtained by approximating counting operations by non-functional fragments of Presburger constraints between the variables of the current state and the variables of the next state. As a consequence of this abstraction, the set of successors of a state is potentially infinite. Examples include the class of Monotonicity Constraint Systems (MCS) [4] and its variants, like constraint automata in [11], and integral relation automata (IRA) [8], for which the (monotonicity) constraints (MC) are boolean combinations of inequalities of the form $u < v$ or $u \leq v$, where u and v range over variables or integer constants. MCS and their subclasses (namely, *size-change systems*) have found important applications for automated termination proofs of functional programs (see e.g. [4]). Richer classes of non-functional fragments of Presburger constraints have been investigated, e.g. difference bound constraints [10], and their extension, namely octagon relations [6], where it is shown that the transitive closure of a single constraint is Presburger definable (these results are useful for the verification of safety properties of flat counter systems).

Recently, an (infinitely-branching) abstract model of counter systems, namely *gap-order constraint systems* (GCS), has been introduced [7], where the constraints (over $\mathbb{Z}$) between the variables of the source state and the target state of a transition are (transitional) *gap-order constraints* (GC) [17]. These constraints are positive boolean combinations of inequalities of the form $u - v \geq k$, where u, v range over variables and integer constants and k is a natural number. Thus, GC can express simple relations on variables such as lower and upper bounds on the values of individual variables; and equality, and gaps (minimal differences) between values of pairs of variables. GC have been introduced in the field of constraint query languages (constraint Datalog) for deductive databases [17], and also have found applications in the analysis of safety properties for parameterized systems [1, 2] and for determining state invariants in counter systems [12]. As pointed out in [2], using GC for expressing the enabling conditions of transitions allow to handle a large class of protocols, where the behavior depends on the relative ordering of values among variables, rather than the actual values of these variables. GCS *strictly* extend MCS (and its variants, namely IRA, and the constraint automata in [11]). This because GC extend MC and, differently from MC, are closed under existential quantification (but not under negation). Hence, GC are closed under composition (which captures the reachability relation for a fixed path in the control graph). Note that if we extend the constraint language of GCS by allowing either negation, or constraints of the form $u - v \geq -k$, with $k \in \mathbb{N}$, then the resulting class of systems can trivially emulate Minsky counter machines, leading to undecidable basic decision problems.

Our contribution. We address termination analysis of GCS. Since GCS are infinitely-branching, termination (i.e., the non-existence of states from which there is an infinite run) does not imply the existence of an upper bound on the lengths of the runs from a given state. The fulfillment of this last condition, we call *strong termination*, can be a necessary requirement in some contexts, such as running-time analysis [3] for infinitely-branching formalisms. Checking usual termination for GCS is known to be decidable and PSPACE-complete [7]. In this paper, by a non-trivial extension of the approach used in [7], we establish the following results:

- (1) For each control location of the given GCS, it is possible to compute a GC representation of the set of variable valuations from which strong termination does *not* hold, and (2) checking strong termination and strong termination for a designated state in GCS are decidable and PSPACE-complete.

Our approach is as follows. First, we consider a subclass of GCS, called *simple* GCS: we establish our first result for simple GCS, and provide a polynomial-time checkable condition for verifying strong termination in simple GCS, which is independent on the size of the lower bounds k in GC. Second, for a given unrestricted GCS $\mathcal{S}$, we show that it is possible to construct a finite family $\mathcal{F}$ of *simple* GCS such that the union of the sets of strongly-terminating states of the single components in $\mathcal{F}$ correspond to the set of strongly-terminating states of $\mathcal{S}$. Then, we show that it is possible to compute separately and in exponential time suitable abstractions of the simple GCS in $\mathcal{F}$ (we are not able to give an upper bound on the size of $\mathcal{F}$), which preserve the fulfillment of the above polynomial-time checkable condition for simple GCS. This leads to exponential-time procedures for solving strong termination and strong termination for a designated state in GCS. Finally, we show that in fact, the considered problems are PSPACE-complete.

A potential application of our results is to use them as basic tool in running-time analysis (based on GCS abstraction) of infinitely-branching computational systems. Note that concurrent open systems are usually infinitely-branching because of the ongoing interaction with an unpredictable environment, and GCS can be used to abstractly model such an interaction. Furthermore, our results can be used to check a restricted class of bounded universal eventually properties in GCS: checking whether for a given state s and control location q, there is a bound k such that each (maximal) computation from s visits location q in less than k steps.

Related work. Strong termination has been addressed in [5]. There, it is shown that for the subclass of MCS where integer constants are disallowed except for 0, checking strong termination is PSPACE-complete. Note that our results extend the above result in two directions: (1) we consider a strict extension of MCS, namely GCS, and (ii) our symbolic algorithm builds a GC representation of the set of non-strongly-terminating states, a very substantial information compared to the algorithm in [5] (see also [4]). For example, by using such a decidable finite representation one can check whether two GCS have the same set of strongly-terminating states.

2 Preliminaries

Let $\mathbb{Z}$ (resp., $\mathbb{N}$) be the set of integers (resp., natural numbers). We fix a finite set $Var = \{x_1, \ldots, x_r\}$ of variables, a finite set of constants $Const \subseteq \mathbb{Z}$ such that $0 \in Const$, and a fresh copy of Var, $Var' = \{x'_1, \ldots, x'_r\}$. For an arbitrary finite set of variables V, an (integer) *valuation* over V is a mapping of the form $\nu : V \to \mathbb{Z}$, assigning to each variable in V an integer value. For $V' \subseteq V$, $\nu_{V'}$ denotes the restriction of ν to V'. For a valuation ν, by convention, we define $\nu(c) = c$ for all $c \in \mathbb{Z}$.

Definition 1. [17] A *gap-order constraint* (GC) over V and $Const$ is a conjunction ξ of inequalities of the form $u - v \geq k$, where $u, v \in V \cup Const$ and $k \in \mathbb{N}$. W.l.o.g. we assume that for all $u, v \in V \cup Const$, there is at most one conjunct in ξ of the form $u - v \geq k$ for some k. A valuation $\nu : V \to \mathbb{Z}$ *satisfies* ξ if for each conjunct $u - v \geq k$ of ξ, $\nu(u) - \nu(v) \geq k$. We denote by $Sat(\xi)$ the set of such valuations.

Definition 2. [8] A (*gap-order*) *monotonicity graph* (MG) over V and $Const$ is a directed weighted graph G with set of vertices $V \cup Const$ and edges $u \xrightarrow{k} v$ labeled by natural numbers k, and s.t.: if $u \xrightarrow{k} v$ and $u \xrightarrow{k'} v$ are edges of G, then $k = k'$. The set $Sat(G)$ of *solutions* of G is the set of valuations ν over V s.t. for each $u \xrightarrow{k} v$ in G, $\nu(u) - \nu(v) \geq k$. GC and MG are equivalent formalisms since there is a trivial linear-time computable bijection assigning to each GC ξ an MG $G(\xi)$ such that $Sat(G(\xi)) = Sat(\xi)$.[1]

The notation $G \models u < v$ means that there is an edge in G from v to u with weight $k > 0$. Moreover, $G \models u \leq v$ means that there is an edge of G from v to u, and $G \models u = v$ means $G \models u \leq v$ and $G \models v \leq u$. Also, we write $G \models u_1 \lhd_1 \ldots \lhd_{n-1} u_n$ to mean that $G \models u_i \lhd_i u_{i+1}$ for each $1 \leq i < n$, where $\lhd_i \in \{<, \leq, =\}$. A *transitional*

[1] MG are called Positive Graphose Inequality Systems in [8].

GC (resp., *transitional* MG) is a GC (resp., MG) over $Var \cup Var'$ and $Const$. For valuations $\nu, \nu' : Var \to \mathbb{Z}$, we denote by $\nu \oplus \nu'$ the valuation over $Var \cup Var'$ defined as follows: $(\nu \oplus \nu')(x_i) = \nu(x_i)$ and $(\nu \oplus \nu')(x'_i) = \nu'(x_i)$ for $i = 1, \ldots, r$.

Definition 3. [7] A *gap-order constraint system* (GCS) over Var and $Const$ is a finite directed labeled graph $\mathcal{S}$ such that each edge is labeled by a *transitional* GC. $Q(\mathcal{S})$ denotes the set of vertices in $\mathcal{S}$, called *control points*, and $E(\mathcal{S})$ the set of edges.

For a finite path $\wp$ of a GCS $\mathcal{S}$, $s(\wp)$ and $t(\wp)$ denote the source and target control points of $\wp$. For a finite path $\wp$ and a path $\wp'$ such that $t(\wp) = s(\wp')$, the composition of $\wp$ and $\wp'$, written $\wp\wp'$, is defined as usual.
The semantics of a GCS $\mathcal{S}$ is given by an infinite directed graph $[\![\mathcal{S}]\!]$ defined as:

- the vertices of $[\![\mathcal{S}]\!]$, called *states* of $\mathcal{S}$, are the pairs of the form (q, ν), where q is a control point of $\mathcal{S}$ and $\nu : Var \to \mathbb{Z}$ is a valuation over Var;
- there is an edge in $[\![\mathcal{S}]\!]$ from (q, ν) to (q', ν') iff there is a (labeled) edge in $\mathcal{S}$ of the form $q \xrightarrow{\xi} q'$ such that $\nu \oplus \nu' \in Sat(\xi)$. We say that the edge of $[\![\mathcal{S}]\!]$ from (q, ν) to (q', ν') is an *instance* of the edge $q \xrightarrow{\xi} q'$ of $\mathcal{S}$.

A path of $[\![\mathcal{S}]\!]$ is called a *run* of $\mathcal{S}$. The length $|\wp|$ (resp., $|\pi|$) of a path $\wp$ (resp., run π) of $\mathcal{S}$ is defined in the standard way. A *non-null* path of $\mathcal{S}$ is a path of $\mathcal{S}$ of non-null length. Let $\wp = q_0 \xrightarrow{\xi_0} q_1 \xrightarrow{\xi_1} q_2, \ldots$ be a path of $\mathcal{S}$. A run π of $\mathcal{S}$ is an *instance* of $\wp$ if π is of the form $\pi = (q_0, \nu_0) \to (q_1, \nu_1) \to (q_2, \nu_2), \ldots$ and for each i, $(q_i, \nu_i) \to (q_{i+1}, \nu_{i+1})$ is an instance of $q_i \xrightarrow{\xi_i} q_{i+1}$. A state s of $\mathcal{S}$ is *terminating* if there is *no* infinite run of $\mathcal{S}$ starting from s. A state s of $\mathcal{S}$ is *unbounded* if the set of lengths of the *finite* runs of $\mathcal{S}$ starting from s is unbounded (equivalently, infinite). A state s of $\mathcal{S}$ is *strongly terminating* if it is *not* unbounded. Since $[\![\mathcal{S}]\!]$ is infinitely-branching, termination and strong termination are distinct concepts. In particular, strong termination implies termination, but the vice-versa in general does not hold.

Example 1. Consider the GCS $\mathcal{S}$ consisting of two self-loops $q \xrightarrow{\xi} q$ and $q \xrightarrow{\xi'} q$, where: $\xi = [(x'_1 < x_1) \wedge (x_1 \geq 0) \wedge (x_2 \geq 0)]$ and $\xi' = [(x'_1 = x_1) \wedge (x'_2 < x_2) \wedge (x_1 \geq 0) \wedge (x_2 \geq 0)]$. Each state of $\mathcal{S}$ is terminating since along any run, the pair (x_1, x_2) decreases strictly w.r.t. the lexicographic order (over $\mathbb{N} \times \mathbb{N}$). On the other hand, one can easily check that each state $s = (q, \nu)$ with $\nu(x_1) > 0$ and $\nu(x_2) \geq 0$ is unbounded.

Since we use MG representations to manipulate GC, we assume that the edge-labels in GCS are transitional MG. A set U of states of a GCS $\mathcal{S}$ is MG *representable* if there is a a family $\{\mathcal{G}_q\}_{q \in Q(\mathcal{S})}$ of finite sets of MG over Var and $Const$ such that $\bigcup_{G \in \mathcal{G}_q} Sat(G) = \{\nu \mid (q, \nu) \in U\}$ for each $q \in Q(\mathcal{S})$.

Investigated problems. For a GCS $\mathcal{S}$, we denote by $Inf_{\mathcal{S}}$ the set of non-terminating states s of $\mathcal{S}$ and by $Unb_{\mathcal{S}}$ the set of unbounded states of $\mathcal{S}$. Note that $Inf_{\mathcal{S}} \subseteq Unb_{\mathcal{S}}$. Moreover, for $q \in Q(\mathcal{S})$, we denote by $Inf^q_{\mathcal{S}}$ (resp., $Unb^q_{\mathcal{S}}$) the set of states in $Inf_{\mathcal{S}}$ (resp., $Unb_{\mathcal{S}}$) of the form (q, ν) for some valuation ν. The termination problem, i.e. checking whether $Inf_{\mathcal{S}} = \emptyset$ for a given GCS $\mathcal{S}$, is known to be decidable and PSPACE-complete [7]. In this paper, we address strong termination:

- *Strong Termination Problem*: given a GCS $\mathcal{S}$, is the set $Unb_{\mathcal{S}}$ empty?

– *Strong Termination Problem w.r.t. a designated state*: given a GCS $\mathcal{S}$ and a state s of $\mathcal{S}$, does $s \notin Unb_{\mathcal{S}}$ hold?

2.1 Properties of monotonicity graphs

We recall some basic properties of MG [8]. Furthermore, we recall a sound and complete (w.r.t. satisfiability) approximation scheme of MG [7] such that basic operations on MG preserve soundness and completeness of this approximation.

A MG G is *satisfiable* if $Sat(G) \neq \emptyset$. Let G be a MG over V and $Const$. For $V' \subseteq V$, *the restriction of G to* V', written $G_{V'}$, is the MG given by the subgraph of G whose set of vertices is $V' \cup Const$. For all vertices u, v of G, we denote by $p_G(u, v)$ the least upper bound (possibly ∞) of the weight sums on all paths in G from u to v (we set $p_G(u, v) = -\infty$ if there is no such a path). The MG G is *normalized* iff: (1) for all vertices u, v of G, if $p_G(u, v) > -\infty$, then $p_G(u, v) \neq \infty$ and $u \overset{p_G(u,v)}{\rightarrow} v$ is an edge of G, and (2) for all constants $c_1, c_2 \in Const$, $p_G(c_1, c_2) \leq c_1 - c_2$.

Proposition 1. [8] *Let G be a* MG *over V and Const. Then:*
1. *If G is* normalized*, then G is satisfiable. Moreover, for all $V' \subseteq V$, every solution of $G_{V'}$ can be extended to a whole solution of G.*
2. *One can check in polynomial time if G is satisfiable. Moreover, if G is satisfiable, then one can build in polynomial time an equivalent* normalized MG $\overline{G}$ *(i.e., $Sat(\overline{G}) = Sat(G)$), called the* closure *of G.*

According to Proposition 1, for a satisfiable MG G, we denote by $\overline{G}$ the closure of G. Moreover, for all unsatisfiable MG G over V and $Const$, we use an unique closure corresponding to some MG G_{nil} over V and $Const$ such that $(G_{nil})_{\emptyset}$ is unsatisfiable. Now, we recall some effective operations on MG. Let $Var'' = \{x''_1, \ldots, x''_r\}$ be an additional copy of $Var = \{x_1 \ldots, x_r\}$.

Definition 4. [8] Let G be a MG on V and $Const$ and G' be a MG on V' and $Const$.
1. **Projection:** if $V' \subseteq V$, the *projection of G over* V' is the MG given by $(\,\overline{G}\,)_{V'}$.
2. **Intersection:** the *intersection* $G \bigotimes G'$ *of G and* G' is the MG over $V \cup V'$ and $Const$ defined as: $u \overset{k}{\rightarrow} v$ is an edge of $G \bigotimes G'$ iff *either* (1) $u \overset{k}{\rightarrow} v$ is an edge of G (resp., G') and there is no edge from u to v in G' (resp., G), *or* (2) $k = max(\{k', k''\})$, $u \overset{k'}{\rightarrow} v$ is an edge of G and $u \overset{k''}{\rightarrow} v$ is an edge of G'.
3. **Composition:** assume that G and G' are two transitional MG. Let G'' be obtained from G' by renaming any variable x'_i into x''_i and x_i into x'_i. The *composition* $G \bullet G'$ of G and G' is the transitional MG obtained from the *projection* of $G \bigotimes G''$ over $Var \cup Var''$ by renaming any variable x''_i into x'_i.

By Definition 4 and Proposition 1, we easily obtain the following known result [8], which essentially asserts that MG (or, equivalently, GC) are closed under intersection and existential quantification.

Proposition 2. *Let G be a* MG *over V and Const and* G' *be a* MG *over* V' *and Const.*
1. ***Projection:*** *if G' is the projection of G over V', then for $\nu' : V' \rightarrow \mathbb{Z}$, $\nu' \in Sat(G')$ iff $\nu' = \nu|_{V'}$ for some $\nu \in Sat(G)$.*

2. ***Intersection:*** *for $\nu : V \cup V' \to \mathbb{Z}$, $\nu \in Sat(G \bigotimes G')$ iff $\nu|_V \in Sat(G)$ and $\nu|_{V'} \in Sat(G')$. Hence, for $V = V'$, $Sat(G \bigotimes G') = Sat(G) \cap Sat(G')$.*
3. ***Composition:*** *assume that G and G' are transitional* MG*. Then, for all $\nu, \nu' : Var \to \mathbb{Z}$. $\nu \oplus \nu' \in Sat(G \bullet G')$ iff $\nu \oplus \nu'' \in Sat(G)$ and $\nu'' \oplus \nu' \in Sat(G')$ for some $\nu'' : Var \to \mathbb{Z}$. Moreover, the composition operator $\bullet$ is associative.*

Approximation scheme: let K stand for $max(\{|c_1 - c_2| + 1 \mid c_1, c_2 \in Const\})$. Note that $K > 0$. For each $h \in \mathbb{N}$, let $\lfloor h \rfloor_K = h$ if $h \leq K$, and $\lfloor h \rfloor_K = K$ otherwise.

Definition 5 (K-bounded MG). [7] A MG is *K-bounded* iff for each of its edges $u \xrightarrow{k} v$, $k \leq K$. For a MG G on V and $Const$, $\lfloor G \rfloor_K$ denotes the K-bounded MG over V and $Const$ obtained from G by replacing each edge $u \xrightarrow{k} v$ of G with the edge $u \xrightarrow{\lfloor k \rfloor_K} v$.

Proposition 3. [7] *Let G be a* MG *over V and $Const$. Then, G is satisfiable iff $\lfloor G \rfloor_K$ is satisfiable. Moreover, $\lfloor \overline{G} \rfloor_K = \lfloor \overline{\lfloor G \rfloor_K} \rfloor_K$. Furthermore, for transitional* MG *G_1 and G_2, $\lfloor G_1 \bullet G_2 \rfloor_K = \lfloor \lfloor G_1 \rfloor_K \bullet \lfloor G_2 \rfloor_K \rfloor_K$.*

2.2 Results on the reachability relation in GCS

We recall some constructive results on the reachability relation in GCS [7].

Definition 6. *A transitional* MG *G is said to be* complete *if:*
- *for all $u, v \in Var \cup Var' \cup Const$, $G \models u \leq v \Rightarrow G \models u \lhd v$ for some $\lhd \in \{<, =\}$;*
- *for all $u, v \in Var \cup Const$, either $G \models u \leq v$ or $G \models v \leq u$;*
- *for all $u, v \in Var' \cup Const$, either $G \models u \leq v$ or $G \models v \leq u$.*

A GCS $\mathcal{S}$ is *complete* iff each MG in $\mathcal{S}$ is complete. Fix a *complete* GCS $\mathcal{S}$. For a finite path $\wp$ of $\mathcal{S}$, the *reachability relation w.r.t.* $\wp$, denoted by $\rightsquigarrow_\wp$, is the binary relation on the set of valuations over Var defined as: for all $\nu, \nu' : Var \to \mathbb{Z}$, $\nu \rightsquigarrow_\wp \nu'$ iff there is a run of $\mathcal{S}$ from $(s(\wp), \nu)$ to $(t(\wp), \nu')$ which is an instance of the path $\wp$. For a transitional MG G, *G characterizes the reachability relation* $\rightsquigarrow_\wp$ iff $Sat(G) = \{\nu \oplus \nu' \mid \nu \rightsquigarrow_\wp \nu'\}$. We associate to each non-null finite path $\wp$ of $\mathcal{S}$ a transitional MG $G_\wp$ and a transitional K-bounded MG $G_\wp^{bd}$, defined by induction on $\wp$ as follows:

- $\wp = q \xrightarrow{G} q'$: $G_\wp = \overline{G}$ and $G_\wp^{bd} = \lfloor \overline{G} \rfloor_K$;
- $\wp = \wp'\wp''$, $|\wp'| > 0$, and $\wp'' = q \xrightarrow{G} q'$: $G_\wp = G_{\wp'} \bullet G$ and $G_\wp^{bd} = \lfloor G_{\wp'}^{bd} \bullet \lfloor G \rfloor_K \rfloor_K$.

The following results have been shown in [7]. Theorem 1 can be easily deduced, while Theorem 2 is a refinement of a result in [8] establishing that for a GCS $\mathcal{S}$, the reflexive transitive closure of the transition relation of $[\![\mathcal{S}]\!]$ is effectively GC definable (a similar result can be found in [17], where it is shown that for Datalog queries with GC, there is a closed form evaluation). Note that in Theorem 2, we are not able to give an upper bound on the cardinality of the set $\mathcal{P}_\mathcal{S}$.

Theorem 1. [7] *For a non-null finite path $\wp$ of $\mathcal{S}$, $G_\wp = \overline{G_\wp}$, $G_\wp^{bd} = \lfloor G_\wp \rfloor_K$ and is complete, and $G_\wp$ is complete and characterizes the reachability relation $\rightsquigarrow_\wp$. Moreover, the set $\{(\lfloor G_\wp \rfloor_K, s(\wp), t(\wp)) \mid \wp$ is a non-null finite path and $G_\wp$ is satisfiable$\}$. has size bounded by $O(|Q(\mathcal{S})|^2 \cdot (K+2)^{(2|Var|+|Const|)^2})$ and can be computed in time $O(|E(\mathcal{S})| \cdot |Q(\mathcal{S})|^2 \cdot (K+2)^{(2|Var|+|Const|)^2})$.*

Theorem 2. [7] *One can compute a* finite *set $\mathcal{P}_\mathcal{S}$ of non-null finite paths of $\mathcal{S}$ such that: for each non-null finite path $\wp'$ of $\mathcal{S}$ from q to q', there is a path $\wp \in \mathcal{P}_\mathcal{S}$ from q to q' so that $\lfloor G_{\wp'} \rfloor_K = \lfloor G_\wp \rfloor_K$, and $\rightsquigarrow_{\wp'}$ implies $\rightsquigarrow_\wp$.*

3 Strong termination for simple GCS

In this section, we solve strong termination for a restricted class of GCS introduced in [7]. A MG G is *weakly normalized* if for all vertices u, v, $p_G(u,v) \geq 0$ (resp., $p_G(u,v) > 0$) implies $G \models v \leq u$ (resp., $G \models v < u$). Note that G is weakly normalized iff $\lfloor G \rfloor_K$ is weakly normalized.
A transitional MG G is *(weakly) idempotent* iff $\lfloor G \bullet G \rfloor_K = \lfloor G \rfloor_K$.

Definition 7 (Simple GCS). [7] *A simple* GCS *is a* GCS *consisting of just two edges of the form $q_0 \xrightarrow{G_0} q$ and $q \xrightarrow{G} q$ such that $q_0 \neq q$. Moreover, we require that $G_0 \bullet G$ is satisfiable, G_0 and G are complete and* weakly *normalized, and G is idempotent.*

To present our results on simple GCS, we need additional definitions.

Definition 8 (lower and upper variables). [7] We denote by MAX (resp., MIN) the maximum (resp., minimum) of $Const$. For a transitional MG G and $y \in Var \cup Var'$, y is a *lower* (resp., *upper*) *variable* of G if $G \models y < MIN$ (resp., $G \models MAX < y$). Moreover, y is a *bounded variable* of G if $G \models MIN \leq y$ and $G \models y \leq MAX$.

Definition 9. A transitional MG is *balanced* iff for all $u, v \in Var \cup Const$ and $\lhd \in \{<, =\}$, $G \models u \lhd v$ iff $G \models u' \lhd v'$ (where for $u \in Var \cup Const$, we write u' to denote the corresponding variable in Var' if $u \in Var$, and u itself otherwise).

Fix a simple GCS $\mathcal{S}$ with edges $q_0 \xrightarrow{G_0} q$ and $q \xrightarrow{G} q$. Since G is idempotent, by the associativity of $\bullet$ and Proposition 3, we obtain that for each $k \geq 1$, $\lfloor G_0 \bullet \underbrace{G \bullet \ldots \bullet G}_{k \text{ times}} \rfloor_K = \lfloor G_0 \bullet G \rfloor_K$. Hence, $G_0 \bullet \underbrace{G \bullet \ldots \bullet G}_{k \text{ times}}$ and $\underbrace{G \bullet \ldots \bullet G}_{k \text{ times}}$ are satisfiable for each $k \geq 1$. Since G is complete, by Proposition 2 it follows that G is *balanced* as well. Moreover, since G is satisfiable and complete, a variable $y \in Var \cup Var'$ is or a lower variable, or an upper variable, or a bounded variable of G, where the "or" is exclusive. We denote by $\mathrm{L}_1, \ldots, \mathrm{L}_N$ (resp., $\mathrm{U}_1, \ldots, \mathrm{U}_M$) the lower (resp., the upper) variables of G in Var, and by $\mathrm{B}_1, \ldots, \mathrm{B}_H$ the bounded variables of G in Var. Hence, we can assume that

$$G \models \mathrm{L}_1 \lhd_2 \ldots \lhd_N \mathrm{L}_N < \mathrm{B}_1 \lhd'_2 \ldots \lhd'_H \mathrm{B}_H < \mathrm{U}_1 \lhd''_2 \ldots \lhd''_M \mathrm{U}_M$$

where $\lhd_2 \ldots \lhd_N, \lhd'_2 \ldots \lhd'_H, \lhd''_2 \ldots \lhd''_M \in \{<, =\}$. Since G is balanced it follows that the lower variables (resp., upper variables) of G in Var' are $\mathrm{L}'_1, \ldots, \mathrm{L}'_N$ (resp., $\mathrm{U}'_1, \ldots, \mathrm{U}'_M$), and the bounded variables of G in Var' are $\mathrm{B}'_1, \ldots, \mathrm{B}'_H$. Moreover,

$$G \models \mathrm{L}'_1 \lhd_2 \ldots \lhd_N \mathrm{L}'_N < \mathrm{B}'_1 \lhd'_2 \ldots \lhd'_H \mathrm{B}'_H < \mathrm{U}'_1 \lhd''_2 \ldots \lhd''_M \mathrm{U}'_M$$

Now, we recall a polynomial-time checkable condition on simple GCS [7].

Definition 10 (termination condition). [7] We say that G satisfies the termination condition iff one of the following holds:

lower variables: either $G \models \mathtt{L}_i < \mathtt{L}'_i$ *for some* $1 \leq i \leq N$,
or $G \models \mathtt{L}_i = \mathtt{L}'_i$ *and* $G \models \mathtt{L}'_j < \mathtt{L}_j$ *for some* $1 \leq i < j \leq N$.
upper variables: either $G \models \mathtt{U}'_i < \mathtt{U}_i$ *for some* $1 \leq i \leq M$,
or $G \models \mathtt{U}_j = \mathtt{U}'_j$ *and* $G \models \mathtt{U}_i < \mathtt{U}'_i$ *for some* $1 \leq i < j \leq M$.

Intuitively, the above condition asserts that either there is a lower (resp., upper) variable of G_{Var} whose value strictly increases (resp., decreases) along each run of $\mathcal{S}$, or there are two lower (resp., upper) variables of G_{Var} such that the absolute value of their difference strictly decreases along each run of $\mathcal{S}$. Let $\mathcal{TC}$ be the class of simple GCS satisfying the termination condition. By Definition 10, we easily obtain the following.

Proposition 4. *If $\mathcal{S} \in \mathcal{TC}$, then $Unb^q_{\mathcal{S}} = \emptyset$ (i.e., the set of unbounded states of $\mathcal{S}$ of the form (q, ν) is empty) and $Inf_{\mathcal{S}} = \emptyset$.*

The following result is known and established in [7].

Theorem 3. [7] *Let $\mathcal{S} \notin \mathcal{TC}$. Then, Inf_S is* MG *representable and one can construct a* MG *representation of Inf_S.*

Moreover, we can show the following non-trivial result (a proof is in Appendix A.2).

Theorem 4. *If $\mathcal{S} \notin \mathcal{TC}$, then $Unb_{\mathcal{S}} = Inf_{\mathcal{S}}$ and $Inf^{q_0}_{\mathcal{S}} \neq \emptyset$.*

3.1 $\mathcal{S}$ satisfies the termination condition

We define a polynomial-time checkable condition on simple GCS which implies the termination condition. We will show that it characterizes the simple GCS such that $Inf_{\mathcal{S}} = \emptyset$ and $Unb_{\mathcal{S}} \neq \emptyset$.

Definition 11 (unboundedness condition). We say that $\mathcal{S}$ *satisfies the unboundedness condition* iff $\mathcal{S} \in \mathcal{TC}$ and **none** of the following properties holds:
lower variables: *there is a lower variable* $\mathtt{L}$ *of* G_0 *in* Var *such that*
– either $G_0 \models \mathtt{L} \leq \mathtt{L}'_i$ *and* $G \models \mathtt{L}_i < \mathtt{L}'_i$ *for some* $1 \leq i \leq N$,
– or $G_0 \models \mathtt{L} \leq \mathtt{L}'_i$, $G \models \mathtt{L}_i = \mathtt{L}'_i$, *and* $G \models \mathtt{L}'_j < \mathtt{L}_j$ *for some* $1 \leq i < j \leq N$.
upper variables: *there is an upper variable* $\mathtt{U}$ *of* G_0 *in* Var *such that*
– either $G_0 \models \mathtt{U}'_i \leq \mathtt{U}$ *and* $G \models \mathtt{U}'_i < \mathtt{U}_i$ *for some* $1 \leq i \leq M$,
– or $G_0 \models \mathtt{U}'_i \leq \mathtt{U}$, $G \models \mathtt{U}_i = \mathtt{U}'_i$, *and* $G \models \mathtt{U}'_j > \mathtt{U}_j$ *for some* $1 \leq j < i \leq M$.

Intuitively, the unboundedness condition implies the termination condition and asserts that: (i) there is no lower (resp., upper) variable of $G_{Var'}$ (or equivalently of G_{Var}) whose value strictly increases (resp., decreases) along each run of $\mathcal{S}$ and at the same time is lower (resp., upper) bounded by a lower (resp., upper) variable of G_0 in Var, (ii) there is no pair of lower (resp., upper) variables of $G_{Var'}$ such that the absolute value of their difference strictly decreases along each run of $\mathcal{S}$, and at the same time is lower (resp., upper) bounded by a lower (resp., upper) variable of G_0 in Var.
Let $\mathcal{UC}$ be the class of simple GCS satisfying the unboundedness condition. Note that $\mathcal{UC} \subseteq \mathcal{TC}$. The following result easily follows from Proposition 4 and Definition 11.

Proposition 5. *If $\mathcal{S} \in \mathcal{TC} \setminus \mathcal{UC}$, then $Unb_{\mathcal{S}} = Inf_{\mathcal{S}} = \emptyset$.*

It remains to consider the case when $\mathcal{S} \in \mathcal{UC}$. We define two integers L and U as follows: L is the smallest $1 \leq i \leq N$ such that $G_0 \models \mathtt{L} \leq \mathtt{L}'_i$ for some lower bound variable $\mathtt{L}$ of G_0 in Var and $G \models \mathtt{L}_i = \mathtt{L}'_i$ (if such a i does not exist, we set $L = N+1$). Finally, U is the greatest $1 \leq i \leq M$ such that $G_0 \models \mathtt{U}'_i \leq \mathtt{U}$ for some upper variable $\mathtt{U}$ of G_0 in Var and $G \models \mathtt{U}'_i = \mathtt{U}_i$ (if such a i does not exist, we set $U = 0$). Note that $1 \leq L \leq N+1$ and $0 \leq U \leq M$. The set of *unconstrained variables* in Var, written Unc, consists of the lower variables $\mathtt{L}_i$ such that $1 \leq i < L$ and the upper variables $\mathtt{U}_j$ such that $U < j \leq M$. We denote by Unc' the corresponding subset in Var'. Since $G_0 \bullet G$ is satisfiable, G_0 is complete, and G is balanced and complete, it follows that the sets of upper (resp., lower) variables of G and G_0 in Var' coincide, and the orderings induced by G and G_0 coincide. By Definitions 11 and 10, if $\mathcal{S} \in \mathcal{UC}$, then either (1) there is an upper variable $\mathtt{U}_j$ such that $G_0 \not\models \mathtt{U}'_j \leq \mathtt{U}$ for each upper variable $\mathtt{U}$ of G_0 in Var, or (2) there is a lower variable $\mathtt{L}_j$ such that $G_0 \not\models \mathtt{L} \leq \mathtt{L}'_j$ for each lower variable $\mathtt{L}$ of G_0 in Var. Thus, by the above considerations, it follows that $Unc \neq \emptyset$ if $\mathcal{S} \in \mathcal{UC}$. The following lemma directly follows from definition of Unc.

Lemma 1. *For a valuation $\nu_0 : Var \to \mathbb{Z}$, the set of valuations $\{\nu_{(Var \setminus Unc)} \mid (q, \nu)$ is reachable from (q_0, ν_0) in $[\![\mathcal{S}]\!]\}$ is finite.*

Let $v_L = \mathtt{L}_L$ if $L < N+1$, and $v_L = MIN$ otherwise. Moreover, let $v_U = \mathtt{U}_U$ if $U > 0$, and $v_U = MAX$ otherwise. For a valuation ν over Var such that $\nu \in Sat(G_{Var})$, we denote by N_ν the natural number defined as follows:

$$N_\nu = min(\{\nu(x) - \nu(v) \mid x \in Unc, v \in Unc \cup \{v_L, v_U\} \text{ and } G \models v < x\} \cup \{\nu(v) - \nu(x) \mid x \in Unc, v \in Unc \cup \{v_L, v_U\} \text{ and } G \models x < v\}).$$

where the minimum of the empty set is 0. Let $\Delta \in \mathbb{N}$ be the maximum of the set of edge weights of G. Now, we give two technical lemmata whose proofs are in Appendix A.3. Lemma 2 ensures the following crucial property: let $\mathcal{S} \in \mathcal{UC}$ and $\pi = (q, \nu_0) \ldots (q, \nu')$ be a run of $\mathcal{S}$ of non-null length such that ν_0 and ν' agree on $Var \setminus Unc$. Then, for all $k \geq 1$ and valuations $\nu \in Sat(G_{Var})$ such that ν and ν_0 agree on $Var \setminus Unc$, if N_ν *is sufficiently large*, then there is also a run of length greater than k from (q, ν) (intuitively, obtained by pumping the pseudo-cycle π).

Lemma 2 (Pumping lemma for unboundedness). *Assume that $\mathcal{S} \in \mathcal{UC}$. Let $\nu : Var \to \mathbb{Z}$ and $\nu' : (Var \setminus Unc) \to \mathbb{Z}$ such that $\lceil \frac{N_\nu}{|Var|+1} \rceil > \Delta$, $\nu \in Sat(G_{Var})$, $\nu' \in Sat(G_{Var \setminus Unc})$, $(\nu_{Var \setminus Unc}) \oplus \nu' \in Sat(G_{(Var \setminus Unc) \cup (Var' \setminus Unc')})$. Then, there exists an extension ν'' of ν' over Var such that the following holds:*

- *$\nu \oplus \nu'' \in Sat(G)$, $\nu'' \in Sat(G_{Var})$, and $N_{\nu''} \geq \lceil \frac{N_\nu}{|Var|+1} \rceil$.*

Lemma 3. *Assume that $\mathcal{S} \in \mathcal{UC}$. Let $\nu_0, \nu : Var \to \mathbb{Z}$ be s.t. $\nu_0 \oplus \nu \in Sat(G_0 \bullet G)$ and $\nu \in Sat(G_{Var})$. Then, the following set is* infinite

$$\{N_{\nu'} \mid \nu_0 \oplus \nu' \in Sat(G_0 \bullet G), \nu' \in Sat(G_{Var}), \text{ and } \nu'_{Var \setminus Unc} = \nu_{Var \setminus Unc}\}$$

By Lemmata 1, 2, and 3, we deduce the following result.

Lemma 4. *Assume that $\mathcal{S} \in \mathcal{UC}$. Then, $(q_0, \nu_0) \in Unb_{\mathcal{S}}$* iff *there is a finite run π of $\mathcal{S}$ starting from (q_0, ν_0) of the form $\pi = (q_0, \nu_0)(q, \nu'_0)(q, \nu) \ldots (q, \nu') \ldots (q, \nu'')$ such that $\nu''_{(Var \setminus Unc)} = \nu'_{(Var \setminus Unc)}$, and the subrun $(q, \nu') \ldots (q, \nu'')$ has non-null length.*

Proof. For the right implication $\Rightarrow$, assume that $(q_0, \nu_0) \in Unb_{\mathcal{S}}$. Hence, the set of lengths of the finite runs from (q_0, ν_0) is infinite. Then, by Lemma 1, Property 1 follows. For the left implication $\Leftarrow$, assume that for a valuation ν_0 over Var, there is a finite run π of $\mathcal{S}$ starting from (q_0, ν_0) of the form $\pi = (q_0, \nu_0)(q, \nu_0')(q, \nu) \ldots (q, \nu') \ldots (q, \nu'')$ such that $\nu''_{(Var\setminus Unc)} = \nu'_{(Var\setminus Unc)}$, and the subrun $(q, \nu') \ldots (q, \nu'')$ has non-null length. Let us consider the prefix of π of length 2 given by $(q_0, \nu_0)(q, \nu_0')(q, \nu)$, and let $S_\nu = \{\overline{\nu} \mid \nu_0 \oplus \overline{\nu} \in Sat(G_0 \bullet G), \overline{\nu} \in Sat(G_{Var}), \text{and } \overline{\nu}_{Var\setminus Unc} = \nu_{Var\setminus Unc}\}$. Since $\nu_0 \oplus \nu \in Sat(G_0 \bullet G)$ and $\nu \in Sat(G_{Var})$, by Lemma 3, the set S_ν is infinite, and the set $Int(S_\nu) = \{N_{\overline{\nu}} \mid \overline{\nu} \in S_\nu\}$ is infinite as well. Let us consider the suffix of π, $(q, \nu) \ldots (q, \nu') \ldots (q, \nu'')$, where $\nu''_{(Var\setminus Unc)} = \nu'_{(Var\setminus Unc)}$. Let $\overline{\nu} \in S_\nu$ and $h \geq 1$. Since $\overline{\nu}_{Var\setminus Unc} = \nu_{Var\setminus Unc}$, Lemma 2 (applied repetitively) ensures that there is $n_h \in \mathbb{N}$ such that if $N_{\overline{\nu}} \geq n_h$, then there is a finite run $\pi_0\pi_1 \ldots \pi_h$ from $(q, \overline{\nu})$ so that

- $\pi_0 = (q, \overline{\nu}) \ldots (q, \nu^1)$ and for $1 \leq i \leq h$, π_i is of the form $(q, \nu^i), \ldots, (q, \nu^{i+1})$, has non-null length, and $\nu^i_{Var\setminus Unc} = \nu^{i+1}_{Var\setminus Unc} = \nu''_{(Var\setminus Unc)} = \nu'_{(Var\setminus Unc)}$.

Since $\nu_0 \oplus \overline{\nu} \in Sat(G_0 \bullet G)$, the run $\pi_0\pi_1 \ldots \pi_h$ can be completed (by adding as prefix a run of length 2 from (q_0, ν_0) to $(q, \overline{\nu})$) into a run starting from (q_0, ν_0) (the whole run has length at least h). Since the set $Int(S_\nu)$ is infinite, for each $h \geq 1$, we can always choose $\overline{\nu} \in S_\nu$ in such a way that the above condition holds. Hence, (q_0, ν_0) is unbounded. This concludes the proof of the lemma. □

Theorem 5. *Assume that $\mathcal{S} \in \mathcal{UC}$. Then, $Unb_{\mathcal{S}}$ is* MG *representable and one can construct a* MG *representation of $Unb_{\mathcal{S}}$.*

Proof. Let $\mathcal{S} \in \mathcal{UC}$. Since, $\mathcal{UC} \subseteq \mathcal{TC}$, by Proposition 4, the set of unbounded states of $\mathcal{S}$ of the form (q, ν) is empty. Hence, it suffices to show that we can construct a finite set $\mathcal{G}_{q_0}$ of MG over Var and $Const$ such that $\bigcup_{G \in \mathcal{G}_{q_0}} Sat(G) = \{\nu \mid (q_0, \nu) \in Unb_{\mathcal{S}}\}$. By Theorem 2, one can compute a *finite* set $\mathcal{P}$ of non-null finite paths of $\mathcal{S}$ from q to q such that for each non-null finite path $\wp'$ of $\mathcal{S}$ from q to q, there is a path $\wp \in \mathcal{P}$ so that $\rightsquigarrow_{\wp'}$ implies $\rightsquigarrow_{\wp}$. Note that given $\wp \in \mathcal{P}$, the transitional MG $G_\wp$ (which characterizes the reachability relation $\rightsquigarrow_\wp$) has the form $\underbrace{G \bullet \ldots \bullet G}_{k \text{ times}}$ for some $k \geq 1$.

Let $G_=$ be the transitional MG corresponding to the GC given by $\displaystyle\bigwedge_{x \in Var\setminus Unc} x' = x$.

Then, $\mathcal{G}_{q_0}$ consists of the MG G' over Var and $Const$ such that $G' = (\overline{G''})_{Var}$, where $G'' = G_0 \bullet G_\wp \bullet (G_{\wp'} \bigotimes G_=)$ for some $\wp, \wp' \in \mathcal{P}$. Note that $\mathcal{G}_{q_0}$ can be effectively computed. Correctness of the construction easily follows from Propositions 1 and 2, and Lemma 4. □

Moreover, we show the following non-trivial result (a proof is in Appendix A.4).

Theorem 6. *If $\mathcal{S} \in \mathcal{UC}$, then $Unb^{q_0}_{\mathcal{S}} \neq \emptyset$.*

By Propositions 4 and 5, and Theorems 3, 4, 5, and 6, we obtain the following result, which provides a straightforward polynomial-time algorithm to check strong termination for simple GCS.

Corollary 1. *For a simple* GCS *$\mathcal{S}$, the following holds:*

– *If $\mathcal{S} \notin \mathcal{TC}$, then $Inf_{\mathcal{S}} = Unb_{\mathcal{S}}$ and $Inf_{\mathcal{S}}^{q_0} = Unb_{\mathcal{S}}^{q_0} \neq \emptyset$;*
– *If $\mathcal{S} \in \mathcal{UC}$ (hence, $\mathcal{S} \in \mathcal{TC}$), then $Inf_{\mathcal{S}} = \emptyset$ and $Unb_{\mathcal{S}}^{q_0} \neq \emptyset$;*
– *If $\mathcal{S} \in \mathcal{TC}$ and $\mathcal{S} \notin \mathcal{UC}$, then $Inf_{\mathcal{S}} = Unb_{\mathcal{S}} = \emptyset$.*

Moreover, one can compute a MG *representation of the sets $Inf_{\mathcal{S}}$ and $Unb_{\mathcal{S}}$.*

4 Strong termination for unrestricted GCS

Fix a GCS $\mathcal{S}$. First, we give a characterization of the set of unbounded states of $\mathcal{S}$. For a non-null finite path $\wp$ of $\mathcal{S}$ such that $s(\wp) = t(\wp)$ (i.e., $\wp$ is cyclic), $(\wp)^\omega$ denotes the infinite path $\wp\wp\ldots$. A infinite path $\wp$ of $\mathcal{S}$ of the form $\wp = \wp'(\wp'')^\omega$ is said to be *ultimately periodic*. A state s of $\mathcal{S}$ is *neatly unbounded* w.r.t. an infinite path $\wp$ of $\mathcal{S}$, if there is a sequence of finite runs $(\pi_n)_{n\in\mathbb{N}}$ of $\mathcal{S}$ starting from s such that $\{|\pi_n| \mid n \in \mathbb{N}\}$ is infinite and for each $n \in \mathbb{N}$, π_n is an instance of the prefix of $\wp$ of length $|\pi_n|$. By using Theorem 2 and Ramsey's Theorem (in its infinite version) [16], we show the following (a proof is in Appendix B.1).

Theorem 7 (Characterization Theorem). *Let $\mathcal{S}$ be a complete* GCS *and $\mathcal{P}_{\mathcal{S}}$ be the finite set of non-null finite paths of $\mathcal{S}$ satisfying Theorem 2. Then, a state s of $\mathcal{S}$ is unbounded* iff *s is neatly unbounded w.r.t. a* ultimately periodic *path $\wp_0 \cdot (\wp)^\omega$ such that $\wp_0, \wp \in \mathcal{P}_{\mathcal{S}}$, $G_{\wp_0} \bullet G_\wp$ is satisfiable, $G_\wp$ is idempotent, and $G_{\wp_0}$ and $G_\wp$ are complete and normalized.*

Let $\mathcal{S}$ be a GCS. We denote by $\lfloor \mathcal{S} \rfloor_K$ the GCS obtained from $\mathcal{S}$ by replacing each edge $q \xrightarrow{G} q'$ of $\mathcal{S}$ with the edge $q \xrightarrow{\lfloor G \rfloor_K} q'$. Note that $\lfloor \mathcal{S} \rfloor_K$ is simple iff $\mathcal{S}$ is simple.

Theorem 8. *Let $\mathcal{S}$ be a* GCS. *Then, $Unb_{\mathcal{S}}$ is* MG *representable and one can construct a* MG *representation of $Unb_{\mathcal{S}}$. Moreover, given $q \in Q(\mathcal{S})$, checking whether $Unb_{\mathcal{S}}^q \neq \emptyset$ is in* PSPACE *and can be done in time $O(|E(\mathcal{S})| \cdot |Q(\mathcal{S})|^2 \cdot (K+2)^{(2|Var|+|Const|)^2})$.*

Proof. We assume that $\mathcal{S}$ is complete (the general case easily follows, and details are given in Appendix B.2). Let $\mathcal{P}_{\mathcal{S}}$ be the computable finite set of non-null finite paths of $\mathcal{S}$ satisfying Theorem 2, and let $\mathcal{F}$ be the finite set of *simple* GCS constructed as: $\mathcal{S}' \in \mathcal{F}$ iff $\mathcal{S}'$ is a simple GCS consisting of two edges of the form $(\natural, s(\wp_0)) \xrightarrow{G_{\wp_0}} t(\wp_0)$ and $s(\wp) \xrightarrow{G_\wp} t(\wp)$ such that $\wp_0, \wp \in \mathcal{P}_{\mathcal{S}}$. By Corollary 1, for each $\mathcal{S}' \in \mathcal{F}$, one can compute a MG representation $\mathcal{G}_{\mathcal{S}',in(\mathcal{S}')}$ of $Unb_{\mathcal{S}'}^{(\natural,in(\mathcal{S}'))}$, where $(\natural, in(\mathcal{S}'))$ is the initial control point of $\mathcal{S}'$. Then, by Theorems 2 and 7, $\{\bigcup_{\{\mathcal{S}'\in\mathcal{F} \mid in(\mathcal{S}')=q\}} \mathcal{G}_{\mathcal{S}',in(\mathcal{S}')}\}_{q\in Q(\mathcal{S})}$ is a MG representation of $Unb_{\mathcal{S}}$. Thus, the first part of the theorem holds.

For the second part of the theorem, let $\mathcal{F}_K$ be the set of GCS $\mathcal{S}'$ such that $\mathcal{S}' = \lfloor \mathcal{S}'' \rfloor_K$ for some $\mathcal{S}'' \in \mathcal{F}$. By Definitions 10 and 11, for a simple GCS $\mathcal{S}''$, $\mathcal{S}'' \in \mathcal{TC}$ (resp., $\mathcal{S}'' \in \mathcal{UC}$) iff $\lfloor \mathcal{S}'' \rfloor_K \in \mathcal{TC}$ (resp., $\lfloor \mathcal{S}'' \rfloor_K \in \mathcal{UC}$). Thus, by Corollary 1, we obtain that for $q \in Q(\mathcal{S})$: $Unb_{\mathcal{S}}^q \neq \emptyset$ iff there is $\mathcal{S}' \in \mathcal{F}_K$ with initial control point $(\natural, q)$ such that either $\mathcal{S}' \notin \mathcal{TC}$ or $\mathcal{S}' \in \mathcal{UC}$. Note that this last condition can be checked in time polynomial in the size of $\mathcal{S}'$. Now, the crucial observation is that $\mathcal{F}_K$ can be computed in exponential time since: (i) by Theorem 2, the set $\{(\lfloor G_\wp \rfloor_K, s(\wp), t(\wp)) \mid \wp \in \mathcal{P}_{\mathcal{S}}$ and $\lfloor G_\wp \rfloor_K$ is satisfiable$\}$ coincides with the set $\mathcal{G}_{\mathcal{S}}^K = \{(\lfloor G_\wp \rfloor_K, s(\wp), t(\wp)) \mid$

$\wp$ is a non-null finite path of $\mathcal{S}$ and $\lfloor G_\wp \rfloor_K$ is satisfiable$\}$, (ii) by Theorem 1, the set $\mathcal{G}_\mathcal{S}^K$ can be computed in time $O(|E(\mathcal{S})| \cdot |Q(\mathcal{S})|^2 \cdot (K+2)^{(2|Var|+|Const|)^2})$. Hence, checking whether $Unb_\mathcal{S}^q \neq \emptyset$ can be done in time $O(|E(\mathcal{S})| \cdot |Q(\mathcal{S})|^2 \cdot (K+2)^{(2|Var|+|Const|)^2})$.

It remains to show that checking whether $Unb_\mathcal{S}^q \neq \emptyset$ can be done in polynomial space. We outline a NPSPACE algorithm. Since NPSPACE=PSPACE (by Savitch's theorem), the result follows. At each step, the nondeterministic algorithm guesses two non-null finite paths $\wp_0$ and $\wp$ of $\mathcal{S}$ such that $s(\wp_0) = q$, and compute the GCS $\mathcal{S}'$ having the edges $(\natural, s(\wp_0)) \stackrel{\lfloor G_{\wp_0} \rfloor_K}{\rightarrow} t(\wp_0)$ and $s(\wp) \stackrel{\lfloor G_\wp \rfloor_K}{\rightarrow} t(\wp)$. The algorithm keeps in memory only the MG $\lfloor G_{\wp_0} \rfloor_K$ and $\lfloor G_\wp \rfloor_K$ associated with the paths $\wp_0$ and $\wp$ generated so far, together with their source and target control points. If the current MG $\mathcal{S}'$ corresponds to a simple MG (hence, $\mathcal{S}' \in \mathcal{F}_K$) such that either $\mathcal{S}' \notin \mathcal{TC}$ or $\mathcal{S}' \in \mathcal{UC}$, then the algorithm terminates with success. Otherwise, the algorithm chooses two edges from control points $t(\wp_0)$ and $t(\wp)$, say $t(\wp_0) \stackrel{G_0}{\rightarrow} q_0$ and $t(\wp) \stackrel{G}{\rightarrow} q$, computes the MG $\lfloor \lfloor G_{\wp_0} \rfloor_K \bullet \lfloor G_0 \rfloor_K \rfloor_K$ and $\lfloor \lfloor G_\wp \rfloor_K \bullet \lfloor G \rfloor_K \rfloor_K$ associated with the currently guessed paths, and re-write the memory by replacing $\lfloor G_{\wp_0} \rfloor_K$ and $\lfloor G_\wp \rfloor_K$ with $\lfloor \lfloor G_{\wp_0} \rfloor_K \bullet \lfloor G_0 \rfloor_K \rfloor_K$ and $\lfloor \lfloor G_\wp \rfloor_K \bullet \lfloor G \rfloor_K \rfloor_K$, and $t(\wp_0)$ and $t(\wp)$ with q_0 and q, and the procedure is repeated. □

Corollary 2. *The strong termination problem and the strong termination problem w.r.t. a designated state are both* PSPACE*-complete.*

Proof. By Theorem 8, strong termination is in PSPACE, and checking whether $Unb_\mathcal{S}^q = \emptyset$ for a given GCS $\mathcal{S}$ and $q \in Q(\mathcal{S})$, is in PSPACE too. By an easy linear-time reduction to this last problem, membership in PSPACE for strong termination w.r.t. a designated state follows as well (for details see Appendix B.3). PSPACE-hardness directly follows from PSPACE-hardness of termination for Boolean Programs [14] and the fact that GCS subsume Boolean Programs (note that for Boolean Programs, which are finitely-branching, strong termination corresponds to termination). □

Additional results: the *bounded universal eventually problem* asks whether for a given GCS $\mathcal{S}$, $\mathcal{S}$-state s, and $q \in Q(\mathcal{S})$, it holds that: there is $k \in \mathbb{N}$, such that for each maximal run π of $\mathcal{S}$ from s, there is a prefix of π of length at most k which visits the control point q. By using Corollary 2, we can show the following (a proof is in Appendix C).

Theorem 9. *The bounded universal eventually problem is decidable.*

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Appendix

A Proofs from Section 3

A.1 Preliminary results

We need some additional definitions and preliminary results. Recall that the fixed transitional MG G labeling the self-loop of the fixed simple GCS $\mathcal{S}$ is complete, balanced, idempotent, satisfiable, and weakly normalized. In the following, for each $n \geq 1$, we denote by G^n the transitional MG $\underbrace{G \bullet \ldots \bullet G}_{n \text{ times}}$. Note that G^n is satisfiable for each $n \geq 1$ (G is idempotent). Moreover, we denote by Δ_n the maximum of the set of edge weights in G^n. First, we recall two technical lemmata established in [7] (Appendix B.1), fundamental to understand the "interaction" between the variables in Var and those in Var' for the fixed transitional MG G.

Lemma 5 (lower variables). *[7] Let $1 \leq i < j \leq N$. Then, the following holds:*

1. *if $G \models \mathrm{L}'_i \leq \mathrm{L}_j$, then* either $G \models \mathrm{L}'_j \leq \mathrm{L}_j$ or $G \models \mathrm{L}'_i \leq \mathrm{L}_{j-1}$*;*
2. *if $G \models \mathrm{L}_i \leq \mathrm{L}'_j$, then* either $G \models \mathrm{L}_j \leq \mathrm{L}'_j$ or $G \models \mathrm{L}_i \leq \mathrm{L}'_{j-1}$.

Lemma 6 (upper variables). *[7] Let $1 \leq i < j \leq M$. Then, the following holds:*

1. *if $G \models \mathrm{U}'_i \leq \mathrm{U}_j$, then* either $G \models \mathrm{U}'_j \leq \mathrm{U}_j$ or $G \models \mathrm{U}'_i \leq \mathrm{U}_{j-1}$*;*
2. *if $G \models \mathrm{U}_i \leq \mathrm{U}'_j$, then* either $G \models \mathrm{U}_j \leq \mathrm{U}'_j$ or $G \models \mathrm{U}_i \leq \mathrm{U}'_{j-1}$.

Definition 12. *Let $\overline{L}, \overline{U} \in \mathbb{N}$. We say that the pair $(\overline{L}, \overline{U})$ is* well-formed *w.r.t. G iff the following conditions are satisfied:*

- *$1 \leq \overline{L} \leq N+1$, and either $\overline{L} = N+1$, or $\overline{L} \leq N$ and $G \models \mathrm{L}_{\overline{L}} = \mathrm{L}'_{\overline{L}}$;*
- *$0 \leq \overline{U} \leq M$, and either $\overline{U} = 0$, or $\overline{U} \geq 1$ and $G \models \mathrm{U}_{\overline{U}} = \mathrm{U}'_{\overline{U}}$;*
- *for each $\overline{L} < k \leq N$, $G \not\models \mathrm{L}'_k < \mathrm{L}_k$ and $G \not\models \mathrm{L}_k < \mathrm{L}'_k$;*
- *for each $1 \leq k < \overline{U}$, $G \not\models \mathrm{U}'_k < \mathrm{U}_k$ and $G \not\models \mathrm{U}_k < \mathrm{U}'_k$.*

Lemma 7 (Separation Lemma). *Let $(\overline{L}, \overline{U})$ be well-formed w.r.t. G. Then:*

Lower Variables: *for all $\overline{L} \leq i, j \leq N$ such that for each h between i and j (i and j included) $G \not\models \mathrm{L}_h = \mathrm{L}'_h$, it holds that $G \not\models \mathrm{L}_i \leq \mathrm{L}'_j$ and $G \not\models \mathrm{L}'_i \leq \mathrm{L}_j$.*

Upper Variables: *for all $1 \leq i, j \leq \overline{U}$ such that for each h between i and j (i and j included) $G \not\models \mathrm{U}_h = \mathrm{U}'_h$, it holds that $G \not\models \mathrm{U}_i \leq \mathrm{U}'_j$ and $G \not\models \mathrm{U}'_i \leq \mathrm{U}_j$.*

Proof. <u>*Lower Variables*</u>. Assume that the result does not hold and derive a contradiction. Then, there are $\overline{L} \leq i, j \leq N$ such that for each h between i and j (i and j included) $G \not\models \mathrm{L}_h = \mathrm{L}'_h$, and either $G \models \mathrm{L}_i \leq \mathrm{L}'_j$ or $G \models \mathrm{L}'_i \leq \mathrm{L}_j$. Assume that $G \models \mathrm{L}'_i \leq \mathrm{L}_j$ (the other case is similar). Then, there are two possibilities:

- $j \leq i$: since $G \models \mathrm{L}_j \leq \mathrm{L}_i$ and $G \models \mathrm{L}'_i \leq \mathrm{L}_j$, it follows that $G \models \mathrm{L}'_i \leq \mathrm{L}_i$. By hypothesis, $G \not\models \mathrm{L}'_i = \mathrm{L}_i$. Since G is complete, we obtain that $G \models \mathrm{L}'_i < \mathrm{L}_i$.

– $j > i$: since $G \models \text{L}'_i \leq \text{L}_j$, by applying repeatedly Condition 1 of Lemma 5, it follows that there is $i \leq h \leq j$ so that $G \models \text{L}'_h \leq \text{L}_h$. By hypothesis, $G \not\models \text{L}'_h = \text{L}_h$. Since G is complete, we obtain $G \models \text{L}'_h < \text{L}_h$.

Thus, in both cases we obtain that $G \models \text{L}'_k < \text{L}_k$ for some $\overline{L} \leq k \leq N$, which is a contradiction since $(\overline{L}, \overline{U})$ is well-formed w.r.t. G. Hence, the result follows.

Upper Variables. This case is analogous to the previous case with the unique difference that now we use Lemma 6. □

Lemma 8. *Let $(\overline{L}, \overline{U})$ be well-formed w.r.t. G, $\widehat{Unc} = \{\text{L}_1, \ldots, \text{L}_{\overline{L}-1}\text{U}_{\overline{U}+1}, \ldots, \text{U}_M\}$, and B (resp., B') be the set of bounded variables of G in Var (resp., in Var'). Moreover, let $n \geq 1$ and $\nu : Var \rightarrow \mathbb{Z}$ be a valuation such that $\nu \in Sat((G^n)_{Var})$, $\nu_B \oplus \nu_B \in Sat((G^n)_{B \cup B'})$, and the following holds:*

(i) *for all $u \in Var \cup Const$ and $v \in Var \setminus B$ s.t. $G^n \models v < u$, $\nu(u) - \nu(v) > \Delta_n$;*
(ii) *for all $u \in Var \cup Const$ and $v \in Var \setminus B$ s.t. $G^n \models u < v$, $\nu(v) - \nu(u) > \Delta_n$.*

Then, $\nu \oplus \nu_{Var \setminus \widehat{Unc}}$ is a solution of the restriction of G^n to $Var \cup (Var' \setminus \widehat{Unc}')$, where $\widehat{Unc}'$ denotes the subset of Var' corresponding to $\widehat{Unc}$.

Proof. Note that $Var \setminus \widehat{Unc} = B \cup A$, where $A = \{\text{L}_i \mid \overline{L} \leq i \leq N\} \cup \{\text{U}_i \mid 1 \leq i \leq \overline{U}\}$. Let A' be the subset of Var' corresponding to A, and let $\nu[Var' \leftarrow Var]$ be the valuation over Var' defined as $\nu[Var' \leftarrow Var](x'_i) = \nu(x_i)$ for each $x_i \in Var$. We need to show that $\nu \oplus \nu_{B \cup A} \in Sat((G^n)_{Var \cup A' \cup B'})$. Since G is idempotent and satisfiable, G^n is satisfiable as well. By hypothesis $\nu_B \oplus \nu_B \in Sat((G^n)_{B \cup B'})$ and $\nu \in Sat((G^n)_{Var})$. Thus, since G^n is balanced (G is balanced and idempotent), by Properties (i) and (ii), we obtain that $\nu[Var' \leftarrow Var] \in Sat((G^n)_{Var'})$. Also, note that for all $u \in Var \setminus B$, $b \in B$, and $\sim \in \{<, =, >\}$, $G \models u \sim b'$ iff $G \models u \sim b$. Since G is idempotent, we obtain that for all $u \in Var \setminus B$, $b \in B$, and $\sim \in \{<, =, >\}$, $G^n \models u \sim b'$ iff $G^n \models u \sim b$. Since $\nu_B \oplus \nu_B \in Sat((G^n)_{B \cup B'})$, by Properties (i) and (ii), it follows that $\nu_{Var} \oplus \nu_B \in Sat((G^n)_{Var \cup B'})$. Thus, it remains to show that $\nu_{Var} \oplus \nu_A \in Sat((G^n)_{Var \cup A'})$. Assume the contrary and derive a contradiction. Since $\nu[Var' \leftarrow Var] \in Sat((G^n)_{Var'})$, there must be $v \in Var$ and $x \in A$ such that one of the following holds:

A: $v \xrightarrow{k} x'$ is an edge of G^n and $\nu(v) - \nu(x) < k$;
B: $x' \xrightarrow{k} v$ is an edge of G^n and $\nu(x) - \nu(v) < k$.

We examine only case A (case B is similar), and show that we obtain a contradiction. Since G is idempotent, by Properties (i) and (ii), the following holds:

Property 1: for all $u \in Var \cup Const$ and $v \in Var \setminus B$ s.t. $G \models v < u$, $\nu(u) - \nu(v) > \Delta_n$;
Property 2: for all $u \in Var \cup Const$ and $v \in Var \setminus B$ s.t. $G \models u < v$, $\nu(v) - \nu(u) > \Delta_n$.

Case A: ($v \xrightarrow{k} x'$ is an edge of G^n and $\nu(v) - \nu(x) < k$, where $v \in Var$ and $x \in A$). By definition of A, there can be two cases:

A.1. $x = \mathrm{L}_i$ for some $\overline{L} \leq i \leq N$. If v is not a lower variable, then $G \models v > \mathrm{L}_i$. By Property 1 above, it follows that $\nu(v) - \nu(\mathrm{L}_i) > \Delta_n \geq k$, and we obtain a contradiction. Hence, $v = \mathrm{L}_j$ for some $1 \leq j \leq N$. Thus, in this case, the hypothesis is as follows: $\mathrm{L}_j \xrightarrow{k} \mathrm{L}'_i$ is an edge of G^n, $\overline{L} \leq i \leq N$, and $\nu(\mathrm{L}_j) - \nu(\mathrm{L}_i) < k$. Since G is idempotent and complete, it holds that

- $G \models \mathrm{L}'_i < \mathrm{L}_j$ if $k > 0$, and $G \models \mathrm{L}'_i = \mathrm{L}_j$ otherwise.

First, assume that $1 \leq j < \overline{L}$. We have that $G \models \mathrm{L}_j \lhd \mathrm{L}_{\overline{L}} \lhd' \mathrm{L}_i$ and $G \models \mathrm{L}'_j \lhd \mathrm{L}'_{\overline{L}} \lhd' \mathrm{L}'_i$ for some $\lhd, \lhd' \in \{<, =\}$. Since $G \models \mathrm{L}_{\overline{L}} = \mathrm{L}'_{\overline{L}}$, it follows that $k = 0$ and $\lhd, \lhd' \in \{=\}$, hence, $G \models \mathrm{L}_i = \mathrm{L}_j$. Since G is idempotent and complete, $G^n \models \mathrm{L}_i = \mathrm{L}_j$ must hold. Thus, since $\nu \in Sat((G^n)_{Var})$, we obtain $\nu(\mathrm{L}_j) - \nu(\mathrm{L}_i) = 0 = k$, which is a contradiction. Therefore, $\overline{L} \leq j \leq N$. Since $G \models \mathrm{L}'_i \leq \mathrm{L}_j$, by Lemma 7 (Separation Lemma), there must be h between i and j such that $G \models \mathrm{L}_h = \mathrm{L}'_h$. There are two cases:

- $i \leq j$. Since $G \models \mathrm{L}_i \leq \mathrm{L}_j$, $G^n \models \mathrm{L}_i \leq \mathrm{L}_j$ must hold (G is idempotent), Thus, if $k = 0$, since $\nu \in Sat((G^n)_{Var})$, then $\nu(\mathrm{L}_j) - \nu(\mathrm{L}_i) \geq 0 = k$, which is a contradiction. Therefore, $k > 0$. Hence, $G \models \mathrm{L}'_i < \mathrm{L}_j$. We have that $G \models \mathrm{L}_i \lhd \mathrm{L}_h \lhd' \mathrm{L}_j$ and $G \models \mathrm{L}'_i \lhd \mathrm{L}'_h \lhd' \mathrm{L}'_j$ for some $\lhd, \lhd' \in \{<, =\}$. Since $G \models \mathrm{L}_h = \mathrm{L}'_h$, it follows that $G \models \mathrm{L}_i < \mathrm{L}_j$. By Property 1 above, we obtain that $\nu(\mathrm{L}_j) - \nu(\mathrm{L}_i) > \Delta_n \geq k$, which is a contradiction.
- $i > j$: We have that $G \models \mathrm{L}_j \lhd \mathrm{L}_h \lhd' \mathrm{L}_i$ and $G \models \mathrm{L}'_j \lhd \mathrm{L}'_h \lhd' \mathrm{L}'_i$ for some $\lhd, \lhd' \in \{<, =\}$. Since $G \models \mathrm{L}_h = \mathrm{L}'_h$ and $G \models \mathrm{L}'_i \leq \mathrm{L}_j$, it follows that $k = 0$ and $\lhd, \lhd' \in \{=\}$, hence, $G \models \mathrm{L}_i = \mathrm{L}_j$. Therefore, $G^n \models \mathrm{L}_i = \mathrm{L}_j$ (G is idempotent). Thus, since $\nu \in Sat((G^n)_{Var})$, $\nu(\mathrm{L}_j) - \nu(\mathrm{L}_i) = 0 = k$, which is a contradiction.

A.2. $x = \mathrm{U}_i$ for some $1 \leq i \leq \overline{U}$. Since $v \xrightarrow{k} \mathrm{U}'_i$ is an edge of G^n and G is idempotent, it follows that v is an upper variable as well, i.e. $v = \mathrm{U}_j$ for some $1 \leq j \leq M$. Thus, in this case, the hypothesis is as follows: $\mathrm{U}_j \xrightarrow{k} \mathrm{U}'_i$ is an edge of G^n, $1 \leq i \leq \overline{U}$, and $\nu(\mathrm{U}_j) - \nu(\mathrm{U}_i) < k$. Moreover,

- $G \models \mathrm{U}'_i < \mathrm{U}_j$ if $k > 0$, and $G \models \mathrm{U}'_i = \mathrm{U}_j$ if $k = 0$.

First, assume that $\overline{U} < j \leq M$. Since $G \models \mathrm{U}_i \leq \mathrm{U}_j$, $G^n \models \mathrm{U}_i \leq \mathrm{U}_j$ holds as well (G is idempotent). If $k = 0$, then since $\nu \in Sat((G^n)_{Var})$, we obtain $\nu(\mathrm{U}_j) - \nu(\mathrm{U}_i) \geq 0 = k$, which is a contradiction. Therefore, $k > 0$. Hence, $G \models \mathrm{U}'_i < \mathrm{U}_j$. We have that $G \models \mathrm{U}_i \lhd \mathrm{U}_{\overline{U}} \lhd' \mathrm{U}_j$ and $G \models \mathrm{U}'_i \lhd \mathrm{U}'_{\overline{U}} \lhd' \mathrm{U}'_j$ for some $\lhd, \lhd' \in \{<, =\}$. Since $G \models \mathrm{U}_{\overline{U}} = \mathrm{U}'_{\overline{U}}$, it follows that $G \models \mathrm{U}_i < \mathrm{U}_j$. By Property 1 above, we obtain that $\nu(\mathrm{U}_j) - \nu(\mathrm{U}_i) > \Delta_n \geq k$, which is a contradiction. Therefore, $1 \leq j \leq \overline{U}$. Since $G \models \mathrm{U}'_i \leq \mathrm{U}_j$, by Lemma 7 (Decomposition Lemma), there must be h between i and j s.t. $G \models \mathrm{U}_h = \mathrm{U}'_h$. There are two cases:

- $i \leq j$. Since $G \models \mathrm{U}_i \leq \mathrm{U}_j$, $G^n \models \mathrm{U}_i \leq \mathrm{U}_j$ holds as well (G is idempotent). Thus, if $k = 0$, then since $\nu \in Sat((G^n)_{Var})$, we obtain $\nu(\mathrm{U}_j) - \nu(\mathrm{U}_i) \geq 0 = k$, which is a contradiction. Therefore, $k > 0$. Hence, $G \models \mathrm{U}'_i < \mathrm{U}_j$. We have that $G \models \mathrm{U}_i \lhd \mathrm{U}_h \lhd' \mathrm{U}_j$ and $G \models \mathrm{U}'_i \lhd \mathrm{U}'_h \lhd' \mathrm{U}'_j$ for some $\lhd, \lhd' \in \{<, =\}$. Since $G \models \mathrm{U}_h = \mathrm{U}'_h$, it follows that $G \models \mathrm{U}_i < \mathrm{U}_j$. By Property 1 above, we obtain that $\nu(\mathrm{U}_j) - \nu(\mathrm{U}_i) > \Delta_n \geq k$, which is a contradiction.

– $i > j$: We have that $G \models \mathrm{U}_j \lhd \mathrm{U}_h \lhd' \mathrm{U}_i$ and $G \models \mathrm{U}'_j \lhd \mathrm{U}'_h \lhd' \mathrm{U}'_i$ for some $\lhd, \lhd' \in \{<, =\}$. Since $G \models \mathrm{U}_h = \mathrm{U}'_h$ and $G \models \mathrm{U}'_i \leq \mathrm{U}_j$, it follows that $G \models \mathrm{U}'_i = \mathrm{U}_j$, $k = 0$, and $\lhd, \lhd' \in \{=\}$. Hence, $G \models \mathrm{U}_i = \mathrm{U}_j$, and $G^n \models \mathrm{U}_i = \mathrm{U}_j$ as well. Thus, since $\nu \in Sat((G^n)_{Var})$, $\nu(\mathrm{U}_j) - \nu(\mathrm{U}_i) = 0 = k$, which is a contradiction.

This concludes the proof of the lemma. □

A.2 Proof of Theorem 4

First, we recall two results established in [7]. As in Subsection 3.1, but with a different meaning, we define two integers L and U as follows: L is the smallest $1 \leq i \leq N$ such that $G \models \mathrm{L}_i = \mathrm{L}'_i$ (if such an i does not exist, we set $L = N + 1$). Finally, U is the greatest $1 \leq i \leq M$ such that $G \models \mathrm{U}_i = \mathrm{U}'_i$ (if such an i does not exist, we set $U = 0$). Note that $1 \leq L \leq N + 1$ and $0 \leq U \leq M$. The set of *unconstrained variables* in Var, written Unc, consists of the lower variables L_i such that $1 \leq i < L$ and the upper variables U_j such that $U < j \leq M$. We denote by Unc' the corresponding subset in Var'. The following two results (Lemma 9 and Lemma 10) are known and established in [7] (note that Lemma 9 is trivial).

Lemma 9. [7] *For a valuation $\nu_0 : Var \to \mathbb{Z}$, the set of valuations $\{\nu_{(Var \setminus Unc)} \mid (q, \nu)$ is reachable from (q, ν_0) in $[\![\mathcal{S}]\!]\}$ is finite.*

Lemma 10. [7] *Let $\mathcal{S} \notin \mathcal{TC}$. Then, $(q, \nu_0) \in Inf_{\mathcal{S}}$* iff *there is a finite run in $\mathcal{S}$ from (q, ν_0) of the form $(q, \nu_0) \ldots (q, \nu) \ldots (q, \nu')$ such that $\nu'_{(Var \setminus Unc)} = \nu_{(Var \setminus Unc)}$ and the subrun $(q, \nu) \ldots (q, \nu')$ has non-null length.*

Corollary 3. *Let $\mathcal{S} \notin \mathcal{TC}$. Then, $Unb_{\mathcal{S}} = Inf_{\mathcal{S}}$.*

Proof. We need to show that each state $s \in Unb_{\mathcal{S}}$ is in $Inf_{\mathcal{S}}$ as well. If s is of the form (q, ν_0), then by Lemma 9 and Lemma 10, the result follows. Otherwise, s is of the form (q_0, ν_0) and there is a run from s reaching a state of the form (q, ν) such that $(q, \nu) \in Unb_{\mathcal{S}}$, hence, the result follows as well. □

Theorem 4. *If $\mathcal{S} \notin \mathcal{TC}$, then $Unb_{\mathcal{S}} = Inf_{\mathcal{S}}$ and $Inf_{\mathcal{S}}^{q_0} \neq \emptyset$.*

Proof. By Corollary 3, $Unb_{\mathcal{S}} = Inf_{\mathcal{S}}$. It remains to show that $Inf_{\mathcal{S}}^{q_0} \neq \emptyset$. By Lemma 10, it suffices to show that there is a finite run of $\mathcal{S}$ of the form $(q_0, \nu_0)(q, \nu'_0)(q, \nu) \ldots (q, \nu') \ldots (q, \nu'')$ such that $\nu''_{Var \setminus Unc} = \nu'_{Var \setminus Unc}$ and the subrun $(q, \nu') \ldots (q, \nu'')$ has non-null length. This is equivalent to the following:

Claim: there are $k \geq 1$, $h \geq 1$ and valuations $\nu_0, \nu, \nu' : Var \to \mathbb{Z}$ such that $(\nu_0 \oplus \nu) \in Sat(G_0 \bullet G^k)$, $(\nu \oplus \nu') \in Sat(G^h)$, and $\nu_{Var \setminus Unc} = \nu'_{Var \setminus Unc}$.

Proof of the claim: Let B be the set of bounded variables of G in Var, and B' be the corresponding set in Var'. Then, the set $\{\nu_B \mid (q, \nu)$ is reachable from some state (q_0, ν_0) in $[\![\mathcal{S}]\!]\}$ is finite. Therefore, since $G_0 \bullet G^k$ is satisfiable for each $k \geq 1$, there are $k \geq 1$, $h \geq 1$ and valuations $\overline{\nu}_0, \overline{\nu}, \overline{\nu}' : Var \to \mathbb{Z}$ such that $(\overline{\nu}_0 \oplus \overline{\nu}) \in Sat(G_0 \bullet G^k)$, $(\overline{\nu} \oplus \overline{\nu}') \in Sat(G^h)$, and $\overline{\nu}_B = \overline{\nu}'_B$. Note that we can assume that $h > 1$.

Now, note that since G is idempotent, the upper (resp., lower, bounded) variables of G^h correspond to the upper (resp., lower, bounded) variables of G, and the upper (resp., lower, bounded) variables of $G_0 \bullet G^k$ in Var' correspond to the upper (resp., lower, bounded) variables of G in Var'.

By definition of the upper (resp., lower, bounded) variables of a satisfiable transitional MG, it easily follows that for each $\delta \geq 1$, there are valuations ν_0 and ν over Var such that the following holds:

1. $\nu_B = \overline{\nu}_B$ and $(\nu_0 \oplus \nu) \in Sat(G_0 \bullet G^k)$;
2. for all $u \in Var \cup Const$ and $v \in Var \setminus B$ such that $G_0 \bullet G^k \models v' < u'$, $\nu(u) - \nu(v) > \delta$;[2]
3. for all $u \in Var \cup Const$ and $v \in Var \setminus B$ such that $G_0 \bullet G^k \models u' < v'$, $\nu(v) - \nu(u) > \delta$;

Note that since G, $G_0 \bullet G^k$, and G^h are complete, we have that for all $\lhd \in \{<, =\}$ and $u, v \in Var \cup Const$, $G_0 \bullet G^k \models v' \lhd u'$ iff $G \models v' \lhd u'$ iff $G^h \models v' \lhd u'$. Since G is balanced, hence G^h is balanced as well, by Properties 2 and 3 above, we obtain:

4. for all $u \in Var \cup Const$ and $v \in Var \setminus B$ s.t. $G^h \models v < u$, $\nu(u) - \nu(v) > \delta$;
5. for all $u \in Var \cup Const$ and $v \in Var \setminus B$ s.t. $G^h \models u < v$, $\nu(v) - \nu(u) > \delta$;

Let $\delta > \Delta_h$ (where Δ_h is the maximum of the set of edge weights in G^h), and let ν and ν_0 satisfying Properties 1–5 above. Since $(\overline{\nu} \oplus \overline{\nu}') \in Sat(G^h)$ and $\overline{\nu}_B = \overline{\nu}'_B$, by Property 1, it follows that $(\nu_B \oplus \nu_B) \in Sat((G^h)_{B \cup B'})$. By Property 1, $(\nu_0 \oplus \nu) \in Sat(G_0 \bullet G^k)$. Moreover, by Properties 4–5, it follows that $\nu \in Sat((G^h)_{Var})$. Since $\mathcal{S} \notin \mathcal{TC}$, by Definition 12 in Appendix A.1, it follows that the pair (L, U) is *well-formed* w.r.t. G. Hence, valuation ν satisfies the hypothesis of Lemma 8 in Appendix A.1, where $(\overline{L}, \overline{U})$ (resp., $\widehat{Unc}$) in the statement of the lemma corresponds to (L, U) (resp., Unc), and n is for h. Hence, we obtain that $\nu \oplus \nu_{Var \setminus Unc} \in Sat((G^h)_{Var \cup (Var' \setminus Unc')})$. Since G^h is normalized (this directly follows from the definition of composition operator and the fact that $h > 1$ and G^h is satisfiable), there is an extension ν' of $\nu_{Var \setminus Unc}$ such that $\nu \oplus \nu' \in Sat(G^h)$. Since $\nu_0 \oplus \nu \in Sat(G_0 \bullet G^k)$, the claim follows. □

A.3 Proofs of Lemmata 2 and 3 in Subsection 3.1

In order to prove Lemmata 2 and 3, we need additional definitions and preliminary results. In the following, L, U, Unc, v_U, and v_L are as in Subsection 3.1.

Lemma 11. *Assume that $\mathcal{S} \in \mathcal{UC}$. Then, the following holds:*

Lower Variables: *let $1 \leq i, j < L$ such that $G_0 \models \mathrm{L} \leq \mathrm{L}'_i$ and $G_0 \models \widetilde{\mathrm{L}} \leq \mathrm{L}'_j$ for some lower variables $\mathrm{L}, \widetilde{\mathrm{L}} \in Var$ of G_0. Then, $G \not\models \mathrm{L}_i \leq \mathrm{L}'_j$.*

Upper Variables: *let $U < i, j, \leq M$ such that $G_0 \models \mathrm{U}'_i \leq \mathrm{U}$ and $G_0 \models \mathrm{U}'_j \leq \widetilde{\mathrm{U}}$ for some upper variables $\mathrm{U}, \widetilde{\mathrm{U}} \in Var$ of G_0. Then, $G \not\models \mathrm{U}'_i \leq \mathrm{U}_j$.*

[2] for $u \in Var \cup Const$, we write u' to denote the corresponding variable in Var' if $u \in Var$, and u itself otherwise.

Proof. **Lower Variables.** Assume that the result does not hold and derive a contradiction. First, observe that since G_0 is complete, we can assume that $\mathrm{L} = \widetilde{\mathrm{L}}$. Hence, $G_0 \models \mathrm{L} \leq \mathrm{L}'_i$, $G_0 \models \mathrm{L} \leq \mathrm{L}'_j$, $1 \leq i, j < L$, and $G \models \mathrm{L}_i \leq \mathrm{L}'_j$. There are two cases:

- $j \leq i$: since $G \models \mathrm{L}'_j \leq \mathrm{L}'_i$ and $G \models \mathrm{L}_i \leq \mathrm{L}'_j$, it follows that $G \models \mathrm{L}_i \leq \mathrm{L}'_i$.
- $j > i$: since $G \models \mathrm{L}_i \leq \mathrm{L}'_j$, by applying repeatedly Condition 2 of Lemma 5 in Appendix A.1, it follows that there is $i \leq h \leq j$ so that $G \models \mathrm{L}_h \leq \mathrm{L}'_h$. Since $G_0 \models \mathrm{L} \leq \mathrm{L}'_i$ and $G_0 \models \mathrm{L}'_i \leq \mathrm{L}'_h$,[3] it follows that $G_0 \models \mathrm{L} \leq \mathrm{L}'_h$.

Thus, in both cases we obtain that $G \models \mathrm{L}_k \leq \mathrm{L}'_k$ and $G_0 \models \mathrm{L} \leq \mathrm{L}'_k$ for some $k < L$. Since G is complete, $G \models \mathrm{L}_k \lhd \mathrm{L}'_k$ for some $\lhd \in \{<, =\}$, which is a contradiction by definition of L and the fact that $\mathcal{S} \in \mathcal{UC}$. Hence, the result follows.

Upper Variables. Assume that the result does not hold and derive a contradiction. First, observe that since G_0 is complete, we can assume that $\mathrm{U} = \widetilde{\mathrm{U}}$. Hence, $G_0 \models \mathrm{U}'_i \leq \mathrm{U}$, $G_0 \models \mathrm{U}'_j \leq \mathrm{U}$, $U < i, j, \leq M$, and $G \models \mathrm{U}'_i \leq \mathrm{U}_j$. There are two cases:

- $j \leq i$: since $G \models \mathrm{U}'_j \leq \mathrm{U}'_i$ and $G \models \mathrm{U}'_i \leq \mathrm{U}_j$, it follows that $G \models \mathrm{U}'_j \leq \mathrm{U}_j$.
- $j > i$: since $G \models \mathrm{U}'_i \leq \mathrm{U}_j$, by applying repeatedly Condition 1 of Lemma 6 in Appendix A.1, it follows that there is $i \leq h \leq j$ so that $G \models \mathrm{U}'_h \leq \mathrm{U}_h$. Since $G_0 \models \mathrm{U}'_j \leq \mathrm{U}$ and $G_0 \models \mathrm{U}'_h \leq \mathrm{U}'_j$, it follows that $G_0 \models \mathrm{U}'_h \leq \mathrm{U}$.

Thus, in both cases we obtain that $G \models \mathrm{U}'_k \leq \mathrm{U}_k$ and $G_0 \models \mathrm{U}'_k \leq \mathrm{U}$ for some $U < k \leq M$. Since G is complete, $G \models \mathrm{U}'_k \lhd \mathrm{U}_k$ for some $\lhd \in \{<, =\}$, which is a contradiction by definition of U and the fact that $\mathcal{S} \in \mathcal{UC}$. Hence, the result follows, which concludes. □

We define two integers $\widetilde{L}$ and $\widetilde{U}$ as follows: $\widetilde{L}$ is the smallest $1 \leq i \leq N$ such that $G_0 \models \mathrm{L} \leq \mathrm{L}'_i$ for some lower bound variable L of G_0 in Var (if such a i does not exist, we set $\widetilde{L} = N + 1$). The integer $\widetilde{U}$ is the greatest $1 \leq i \leq M$ such that $G_0 \models \mathrm{U}'_i \leq \mathrm{U}$ for some upper variable U of G_0 in Var (if such a i does not exist, we set $\widetilde{U} = 0$). Note that $1 \leq \widetilde{L} \leq L$ and $U \leq \widetilde{U} \leq M$. Let $\widetilde{Unc}$ be the set consisting of the lower variables L_i such that $1 \leq i < \widetilde{L}$ and the upper variables U_j such that $\widetilde{U} < j \leq M$. We denote by $\widetilde{Unc}'$ the corresponding subset in Var'. Note that $\widetilde{Unc} \subseteq Unc$. Let $v_{\widetilde{L}} = \mathrm{L}_{\widetilde{L}}$ if $\widetilde{L} < N + 1$, and $v_{\widetilde{L}} = MIN$ otherwise. Moreover, let $v_{\widetilde{U}} = \mathrm{U}_{\widetilde{U}}$ if $\widetilde{U} > 0$, and $v_{\widetilde{U}} = MAX$ otherwise. For a valuation ν over Var such that $\nu \in Sat(G_{Var})$, we denote by $\widetilde{N}_\nu$ the natural number defined as follows:

$$\widetilde{N}_\nu = min(\{\nu(x) - \nu(v) \mid x \in \widetilde{Unc}, v \in \widetilde{Unc} \cup \{v_{\widetilde{L}}, v_{\widetilde{U}}\} \text{ and } G \models v < x\} \cup \{\nu(v) - \nu(x) \mid x \in \widetilde{Unc}, v \in \widetilde{Unc} \cup \{v_{\widetilde{L}}, v_{\widetilde{U}}\} \text{ and } G \models x < v\}).$$

where the minimum of the empty set is 0. Since $G_0 \bullet G$ is satisfiable, G_0 is complete, and G is balanced and complete, it follows that the sets of upper (resp., lower) variables of G and G_0 in Var' coincide, and the orderings induced by G and G_0 coincide. By Definitions 11 and 10, if $\mathcal{S} \in \mathcal{UC}$, then either (1) there is an upper variable U_j such that $G_0 \not\models \mathrm{U}'_j \leq \mathrm{U}$ for each upper variable U of G_0 in Var, or (2) there is a lower variable

[3] Since $G_0 \bullet G$ is satisfiable, G_0 is complete, and G is complete and balanced, for all $\lhd \in \{<,=\}$, $1 \leq i, j \leq N$, and $1 \leq h, k \leq M$: (1) $G_0 \models \mathrm{L}'_i \lhd \mathrm{L}'_j$ iff $G \models \mathrm{L}'_i \lhd \mathrm{L}'_j$, and (2) $G_0 \models \mathrm{U}'_i \lhd \mathrm{U}'_j$ iff $G \models \mathrm{U}'_i \lhd \mathrm{U}'_j$

L_j such that $G_0 \not\models \text{L} \leq \text{L}'_j$ for each lower variable L of G_0 in Var. Thus, by the above considerations, it follows that $\widetilde{Unc} \neq \emptyset$ if $\mathcal{S} \in \mathcal{UC}$. Since $\widetilde{Unc} \subseteq Unc$, by definition of $\widetilde{N}_\nu$ and N_ν (in Subsection 3.1), we also obtain that $\widetilde{N}_\nu \geq N_\nu$ if $\mathcal{S} \in \mathcal{UC}$.

Since no upper (resp., lower) variable in $\widetilde{Unc}'$ is upper (resp., lower) bounded in G_0 by an upper (resp., lower variable) of G_0 in Var, we easily obtain the following result.

Lemma 12. *Assume that $\mathcal{S} \in \mathcal{UC}$. Let $\nu, \nu' : Var \rightarrow \mathbb{Z}$ with $\nu \oplus \nu' \in Sat(G_0)$ and $\nu' \in Sat(G_{Var})$. Then, the following set is* infinite

$$\{\widetilde{N}_{\nu''} \mid \nu \oplus \nu'' \in Sat(G_0),\ \nu''_{Var \setminus \widetilde{Unc}} = \nu'_{Var \setminus \widetilde{Unc}},\ \textit{and } \nu'' \in Sat(G_{Var})\}$$

Lemma 13. *Assume that $\mathcal{S} \in \mathcal{UC}$. Let $\nu : Var \rightarrow \mathbb{Z}$ and $\nu' : (Var \setminus Unc) \rightarrow \mathbb{Z}$ such that $\lceil \frac{\widetilde{N}_\nu}{|Var|+1} \rceil > \Delta$, $\nu \in Sat(G_{Var})$, $\nu' \in Sat(G_{Var \setminus Unc})$, and $(\nu_{Var \setminus \widetilde{Unc}}) \oplus \nu' \in Sat(G_{(Var \setminus \widetilde{Unc}) \cup (Var' \setminus Unc')})$. Then, there exists an extension ν'' of ν' over Var such that the following holds:*

- *$\nu \oplus \nu'' \in Sat(G)$, $\nu'' \in Sat(G_{Var})$, and $N_{\nu''} \geq \lceil \frac{\widetilde{N}_\nu}{|Var|+1} \rceil$.*

Proof. Let ν and ν' as in the statement of the lemma. Assume that $L \neq N+1$ and $U \neq 0$ (the other cases being similar). Then, $v_U = \text{U}_U$, $v_{\widetilde{U}} = \text{U}_{\widetilde{U}}$, $v_L = \text{L}_L$, $v_{\widetilde{L}} = \text{L}_{\widetilde{L}}$. The following properties directly follow from the definitions of the integers $L, U, \widetilde{L}, \widetilde{U}$, and the sets Unc and $\widetilde{Unc}$.

- **Property 1:** $\widetilde{L} \leq L$, $\widetilde{U} \geq U$, $Unc = \{\text{L}_1, \ldots, \text{L}_{L-1}, \text{U}_{U+1}, \ldots, \text{U}_M\}$, and $\widetilde{Unc} = \{\text{L}_1, \ldots, \text{L}_{\widetilde{L}-1}, \text{U}_{\widetilde{U}+1}, \ldots, \text{U}_M\}$ (in particular, $\widetilde{Unc} \subseteq Unc$);
- **Property 2:** $G \models \text{U}_U = \text{U}'_U$, $G \models \text{L}_L = \text{L}'_L$, and $G \models \text{L}_L \leq x \leq \text{U}_U$ and $G \models \text{L}_L \leq x' \leq \text{U}_U$ for each $x \in Var \setminus Unc$;
- **Property 3:** for each $\text{U}_i \in Unc$, $G \models \text{U}_i > \text{U}_U$ and $G \models \text{U}'_i > \text{U}_U$;
- **Property 4:** for each $\text{L}_i \in Unc$, $G \models \text{L}_i < \text{L}_L$, and $G \models \text{L}'_i < \text{L}_L$.

By hypothesis, definition of Δ and $\widetilde{N}_\nu$, and Properties 1–4 above, it follows that $\nu \oplus \nu'$ is a solution of $G_{Var \cup (Var' \setminus Unc')}$.

By Lemma 11, for all $U < i, j \leq \widetilde{U}$, $G \not\models \text{U}'_j \leq \text{U}_i$, and for all $\widetilde{L} \leq i, j < L$, $G \not\models \text{L}_j \leq \text{L}'_i$. It follows that the upper variables in Unc' are not upper bounded by the upper variables in $Unc \setminus \widetilde{Unc}$, and the lower variables in Unc' are not lower bounded by the lower variables in $Unc \setminus \widetilde{Unc}$. Hence, since G is satisfiable, there exists a linear ordering of the set of vertices $Unc \cup Unc' \cup \{\text{U}_U, \text{L}_L\}$ which is consistent with the constraints $\text{U}_{\widetilde{U}} < \text{U}'_{U+1} \leq \ldots \leq \text{U}'_M$ and $\text{L}'_1 \leq \ldots \leq \text{L}'_{L-1} < \text{L}_{\widetilde{L}}$ and with the constraints of G (in particular, $G \models \text{U}_{\widetilde{U}} < \text{U}_{\widetilde{U}+1} \leq \ldots \leq \text{U}_M$ and $G \models \text{L}_1 \leq \ldots \leq \text{L}_{\widetilde{L}-1} < \text{L}_{\widetilde{L}}$). Moreover, for two consecutive vertices $u, v \in \{\text{U}_{\widetilde{U}}, \text{U}_{\widetilde{U}+1}, \ldots, \text{U}_M\}$ (resp., $u, v \in \{\text{L}_1, \ldots, \text{L}_{\widetilde{L}-1}, \text{L}_{\widetilde{L}}\}$), with $G \not\models u = v$, the maximum number of variables in Unc' which lie (according to the above linear ordering) between u and v is at most $|Var|$, and $|\nu(u) - \nu(v)| \geq \widetilde{N}_\nu$. Then, since $\lceil \frac{\widetilde{N}_\nu}{|Var|+1} \rceil > \Delta$, it is evident that we can assign to the variables in Unc' integers values in such a way that the corresponding extension ν'' of ν' satisfies the statement of the lemma. □

Now, we can prove Lemmata 2 and 3.

Lemma 2. *Assume that $\mathcal{S} \in \mathcal{UC}$. Let $\nu : Var \to \mathbb{Z}$ and $\nu' : (Var \setminus Unc) \to \mathbb{Z}$ such that $\lceil \frac{N_\nu}{|Var|+1} \rceil > \Delta$, $\nu \in Sat(G_{Var})$, $\nu' \in Sat(G_{Var \setminus Unc})$, $(\nu_{Var \setminus Unc}) \oplus \nu' \in Sat(G_{(Var \setminus Unc) \cup (Var' \setminus Unc')})$. Then, there exists an extension ν'' of ν' over Var such that the following holds:*

- $\nu \oplus \nu'' \in Sat(G)$, $\nu'' \in Sat(G_{Var})$, *and* $N_{\nu''} \geq \lceil \frac{N_\nu}{|Var|+1} \rceil$.

Proof. By hypothesis, definition of N_ν, and Properties 1–4 in the proof of Lemma 13, it easily follows that $(\nu_{Var \setminus \widetilde{Unc}}) \oplus \nu' \in Sat(G_{(Var \setminus \widetilde{Unc}) \cup (Var' \setminus Unc')})$. Since $\widetilde{N}_\nu \geq N_\nu$, we can apply Lemma 13. Hence, there exists an extension ν'' of ν' over Var such that the following holds:

- $\nu'' \in Sat(G_{Var})$, $\nu \oplus \nu'' \in Sat(G)$, and $N_{\nu''} \geq \lceil \frac{\widetilde{N}_\nu}{|Var|+1} \rceil$.

Since $\widetilde{N}_\nu \geq N_\nu$, the result follows. □

Lemma 3. *Assume that $\mathcal{S} \in \mathcal{UC}$. Let $\nu_0, \nu : Var \to \mathbb{Z}$ be s.t. $\nu_0 \oplus \nu \in Sat(G_0 \bullet G)$ and $\nu \in Sat(G_{Var})$. Then, the following set is* infinite

$$\{N_{\nu'} \mid \nu_0 \oplus \nu' \in Sat(G_0 \bullet G),\ \nu' \in Sat(G_{Var}),\ \text{and } \nu'_{Var \setminus Unc} = \nu_{Var \setminus Unc}\}$$

Proof. Let $\nu_0, \nu : Var \to \mathbb{Z}$ be such that $\nu_0 \oplus \nu \in Sat(G_0 \bullet G)$ and $\nu \in Sat(G_{Var})$. Then, there is $\nu' : Var \to \mathbb{Z}$ such that $\nu_0 \oplus \nu' \in Sat(G_0)$ and $\nu' \oplus \nu \in Sat(G)$. In particular, $\nu' \in Sat(G_{Var})$. Let us consider the set of valuations $S_{\nu'} = \{\overline{\nu} \mid \nu_0 \oplus \overline{\nu} \in Sat(G_0), \overline{\nu}_{Var \setminus \widetilde{Unc}} = \nu'_{Var \setminus \widetilde{Unc}}, \text{ and } \overline{\nu} \in Sat(G_{Var})\}$. By Lemma 12, the set $S_{\nu'}$ is infinite, and the set $Int(S_{\nu'}) = \{\widetilde{N}_{\overline{\nu}} \mid \overline{\nu} \in S_{\nu'}\}$ is infinite as well. Therefore, the result directly follows from the following claim:

Claim: for each $\overline{\nu} \in S_{\nu'}$ such that $\lceil \frac{\widetilde{N}_{\overline{\nu}}}{|Var|+1} \rceil > \Delta$, there exists an extension ν'' of $\nu_{Var \setminus Unc}$ such that $\nu'' \in Sat(G_{Var})$, $\overline{\nu} \oplus \nu'' \in Sat(G)$, and $N_{\nu''} \geq \lceil \frac{\widetilde{N}_{\overline{\nu}}}{|Var|+1} \rceil$.

Proof of the claim: by hypothesis $\overline{\nu} \in Sat(G_{Var})$, $\lceil \frac{\widetilde{N}_{\overline{\nu}}}{|Var|+1} \rceil > \Delta$, and $\nu \in Sat(G_{Var})$ (hence, in particular, $\nu_{Var \setminus Unc} \in Sat(G_{Var \setminus Unc})$). Since $\overline{\nu}_{Var \setminus \widetilde{Unc}} = \nu'_{Var \setminus \widetilde{Unc}}$ and $\nu' \oplus \nu \in Sat(G)$, $(\overline{\nu}_{Var \setminus \widetilde{Unc}}) \oplus (\nu_{Var \setminus Unc}) \in Sat(G_{(Var \setminus \widetilde{Unc}) \cup (Var' \setminus Unc')})$. Thus, by Lemma 13 there exists an extension ν'' of $\nu_{Var \setminus Unc}$ over Var such that $\nu'' \in Sat(G_{Var})$, $\overline{\nu} \oplus \nu'' \in Sat(G)$, and $N_{\nu''} \geq \lceil \frac{\widetilde{N}_{\overline{\nu}}}{|Var|+1} \rceil$, and we are done. □

A.4 Proof of Theorem 6 in Subsection 3.1

Let L, U, and Unc as defined in Subsection 3.1.

Theorem 6. *If $\mathcal{S} \in \mathcal{UC}$, then $Unb^{q_0}_{\mathcal{S}} \neq \emptyset$.*

Proof. By Lemma 4, it suffices to show that there is a finite run of $\mathcal{S}$ of the form $(q_0, \nu_0)(q, \nu'_0)(q, \nu) \ldots (q, \nu') \ldots (q, \nu'')$ such that $\nu''_{Var \setminus Unc} = \nu'_{Var \setminus Unc}$ and the sub-run $(q, \nu') \ldots (q, \nu'')$ has non-null length. This is equivalent to the following:

Claim: there are $k \geq 1$, $h \geq 1$ and valuations $\nu_0, \nu, \nu' : Var \to \mathbb{Z}$ such that $(\nu_0 \oplus \nu) \in Sat(G_0 \bullet G^k)$, $(\nu \oplus \nu') \in Sat(G^h)$, and $\nu_{Var \setminus Unc} = \nu'_{Var \setminus Unc}$.

Proof of the claim: Let B be the set of bounded variables of G in Var, and B' be the corresponding set in Var'. Then, the set $\{\nu_B \mid (q, \nu)$ is reachable from some state (q_0, ν_0) in $[\![\mathcal{S}]\!]\}$ is finite. Therefore, since $G_0 \bullet G^k$ is satisfiable for each $k \geq 1$, there are $k \geq 1$, $h \geq 1$ and valuations $\overline{\nu}_0, \overline{\nu}, \overline{\nu}' : Var \to \mathbb{Z}$ such that $(\overline{\nu}_0 \oplus \overline{\nu}) \in Sat(G_0 \bullet G^k)$, $(\overline{\nu} \oplus \overline{\nu}') \in Sat(G^h)$, and $\overline{\nu}_B = \overline{\nu}'_B$. Note that we can assume that $h > 1$.

Now, note that since G is idempotent, the upper (resp., lower, bounded) variables of G^h correspond to the upper (resp., lower, bounded) variables of G, and the upper (resp., lower, bounded) variables of $G_0 \bullet G^k$ in Var' correspond to the upper (resp., lower, bounded) variables of G in Var'.

By definition of the upper (resp., lower, bounded) variables of a satisfiable transitional MG, it easily follows that for each $\delta \geq 1$, there are valuations ν_0 and ν over Var such that the following holds:

1. $\nu_B = \overline{\nu}_B$ and $(\nu_0 \oplus \nu) \in Sat(G_0 \bullet G^k)$;
2. for all $u \in Var \cup Const$ and $v \in Var \setminus B$ such that $G_0 \bullet G^k \models v' < u'$, $\nu(u) - \nu(v) > \delta$;[4]
3. for all $u \in Var \cup Const$ and $v \in Var \setminus B$ such that $G_0 \bullet G^k \models u' < v'$, $\nu(v) - \nu(u) > \delta$;

Note that since G, $G_0 \bullet G^k$, and G^h are complete, we have that for all $\lhd \in \{<, =\}$ and $u, v \in Var \cup Const$, $G_0 \bullet G^k \models v' \lhd u'$ iff $G \models v' \lhd u'$ iff $G^h \models v' \lhd u'$. Since G is balanced, hence G_h is balanced as well, by Properties 2 and 3 above, we obtain:

4. for all $u \in Var \cup Const$ and $v \in Var \setminus B$ s.t. $G^h \models v < u$, $\nu(u) - \nu(v) > \delta$;
5. for all $u \in Var \cup Const$ and $v \in Var \setminus B$ s.t. $G^h \models u < v$, $\nu(v) - \nu(u) > \delta$;

Let $\delta > \Delta_h$ (where Δ_h is the maximum of the set of edge weights in G^h), and let ν and ν_0 satisfying Properties 1–5 above. Since $(\overline{\nu} \oplus \overline{\nu}') \in Sat(G^h)$ and $\overline{\nu}_B = \overline{\nu}'_B$, by Property 1, it follows that $(\nu_B \oplus \nu_B) \in Sat((G^h)_{B \cup B'})$. By Property 1, $(\nu_0 \oplus \nu) \in Sat(G_0 \bullet G^k)$. Moreover, by Properties 4–5, it follows that $\nu \in Sat((G^h)_{Var})$. Since $\mathcal{S} \in \mathcal{UC}$, by Definition 12 in Appendix A.1, it follows that the pair (L, U) is *well-formed* w.r.t. G. Hence, valuation ν satisfies the hypothesis of Lemma 8 in Appendix A.1, where $(\overline{L}, \overline{U})$ (resp., $\widehat{Unc}$) in the statement of the lemma corresponds to (L, U) (resp., Unc), and n is for h. Hence, we obtain that $\nu \oplus \nu_{Var \setminus Unc} \in Sat((G^h)_{Var \cup (Var' \setminus Unc')})$. Since G^h is normalized (this directly follows from the definition of composition operator and the fact that $h > 1$ and G^h is satisfiable), there is an extension ν' of $\nu_{Var \setminus Unc}$ such that $\nu \oplus \nu' \in Sat(G^h)$. Since $\nu_0 \oplus \nu \in Sat(G_0 \bullet G^k)$, the claim follows. □

B Proofs from Section 4

B.1 Proof of Theorem 7

For a sequence of runs $(\pi_n)_{n \in \mathbb{N}}$ of a GCS $\mathcal{S}$, a *subsequence* of $(\pi_n)_{n \in \mathbb{N}}$ is a sequence of the form $(\pi_{h_n})_{n \in \mathbb{N}}$ such that $h_n < h_{n+1}$ for each $n \in \mathbb{N}$. First, we show the following.

[4] for $u \in Var \cup Const$, we write u' to denote the corresponding variable in Var' if $u \in Var$, and u itself otherwise.

Lemma 14. *An unbounded state of a* GCS $\mathcal{S}$ *is neatly unbounded (w.r.t. some infinite path of* $\mathcal{S}$*) too.*

Proof. Let s be a unbounded state of $\mathcal{S}$. Hence, there is a sequence of finite runs $(\pi_n)_{n\in\mathbb{N}}$ of $\mathcal{S}$ starting from s such that the set $\{|\pi_n| \mid n \in \mathbb{N}\}$ is infinite. W.l.o.g. we can assume that $|\pi_n| < |\pi_m|$ for $n < m$. First, we note the following:

Claim. For each $h \geq 1$, there is a subsequence $(\pi_n^h)_{n\in\mathbb{N}}$ of $(\pi_n)_{n\in\mathbb{N}}$ and a finite path $\wp_h$ of $\mathcal{S}$ of length h such that the following holds:

- for each $n \in \mathbb{N}$, $|\pi_n^h| \geq h$ and the prefix of π_n^h of length h is an instance of $\wp_h$;
- $(\pi_n^{h+1})_{n\in\mathbb{N}}$ is a subsequence of $(\pi_n^h)_{n\in\mathbb{N}}$;
- $\wp_h$ is a prefix of $\wp_{h+1}$.

Proof of the claim. By a straightforward induction on h (by using the facts that $|\pi_n| < |\pi_m|$ for $n < m$, and the set of paths of $\mathcal{S}$ of length 1 is finite). □

For each $h \geq 1$, let $(\pi_n^h)_{n\in\mathbb{N}}$ and $\wp_h$ as in the claim above. Then, the sequence $(\wp_h)_{h\geq 1}$ corresponds to an infinite path $\wp$ of $\mathcal{S}$, where $\wp_h$ is its prefix of length h. Moreover, it follows that there is a subsequence $(\pi'_n)_{n\in\mathbb{N}}$ of $(\pi_n)_{n\in\mathbb{N}}$ such that for each $n \geq 1$, $|\pi'_n| \geq n$ and the prefix of π'_n of length n is an instance of $\wp_n$. Hence, state s is neatly unbounded w.r.t. the infinite path $\wp$, which concludes. □

Now, we prove Theorem 7.

Theorem 7 (Characterization Theorem). *Let* $\mathcal{S}$ *be a complete* GCS *and* $\mathcal{P}_\mathcal{S}$ *be the finite set of non-null finite paths of* $\mathcal{S}$ *satisfying Theorem 2. Then, a state* s *of* $\mathcal{S}$ *is unbounded* iff s *is neatly unbounded w.r.t. a* ultimately periodic *path* $\wp_0 \cdot (\wp)^\omega$ *such that* $\wp_0, \wp \in \mathcal{P}_\mathcal{S}$*,* $G_{\wp_0} \bullet G_\wp$ *is satisfiable,* $G_\wp$ *is idempotent, and* $G_{\wp_0}$ *and* $G_\wp$ *are complete and normalized.*

Proof. The left implication $\Leftarrow$ is obvious. For the right implication $\Rightarrow$, assume that s is a unbounded state of $\mathcal{S}$. By Lemma 14, there is a sequence $(\pi_n)_{n\in\mathbb{N}}$ of finite runs of $\mathcal{S}$ and an infinite path $\wp_\infty$ of $\mathcal{S}$ such that:

1. for each $h \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that $|\pi_n| > h$;
2. for each $n \in \mathbb{N}$, π_n is an instance of the prefix of $\wp_\infty$ of length $|\pi_n|$.

Let us consider the finite set $\mathcal{P}_\mathcal{S}$ of non-null finite paths of $\mathcal{S}$ satisfying the statement of Theorem 2. For each $\wp \in \mathcal{P}_\mathcal{S}$ from q to q', we denote by $[\wp]$ the set of non-null finite paths $\wp'$ of $\mathcal{S}$ from q to q' such that $\rightsquigarrow_{\wp'}$ implies $\rightsquigarrow_\wp$, and $\lfloor G_{\wp'} \rfloor_K = \lfloor G_\wp \rfloor_K$. Let H be the *finite* set given by $H = \{[\wp] \mid \wp \in \mathcal{P}_\mathcal{S}\}$. For each non-null finite path $\wp'$ of $\mathcal{S}$, we associate to $\wp'$ a *color* given by some element $[\wp] \in H$ such that $\wp' \in [\wp]$ (note that such an element of H must exist). Let us consider the infinite path $\wp_\infty$. Then, there is a control point q such that $\wp_\infty$ is of the form $\wp_\infty = \wp_0\wp_1\wp_2\ldots$, where for each $i \geq 1$, $\wp_i$ is a non-null (cyclic) path from q to q. Let us consider the set of positive natural numbers, and label each pair (i, j) of its elements with $i < j$ with the color of the subpath $\wp_i \ldots \wp_j$ of $\wp_\infty$. By Ramsey's Theorem (in its infinite version)[16], there is an infinite set I of positive natural numbers such that all the pairs (i, j) with $i, j \in I$ (and $i < j$) carry the same label in H, say $[\wp]$. It follows that $\wp_\infty$ can be written in the form $\wp_\infty = \wp'_0\wp'_1\wp'_2\ldots$ such that $|\wp'_0| > 0$ and for all $i \geq 1$, $\wp'_i \in [\wp]$ and $\wp'_i\wp'_{i+1} \in [\wp]$.

Hence, in particular, $\lfloor G_{\wp'_i} \rfloor_K = \lfloor G_\wp \rfloor_K$ and $\lfloor G_{\wp'_i \wp'_{i+1}} \rfloor_K = \lfloor G_\wp \rfloor_K$. By Proposition 3 and associativity of $\bullet$, we obtain that $\lfloor G_\wp \rfloor_K = \lfloor G_\wp \bullet G_\wp \rfloor_K$. Hence, $G_\wp$ is idempotent.

By Conditions 1 and 2 above, there is a sequence of finite runs $(\pi'_n)_{n \geq 0}$ from s such that π'_n is an instance of the prefix $\wp'_0 \wp'_1 \ldots \wp'_n$ of $\wp_\infty$. Let $\wp''_0 \in \mathcal{P}_\mathcal{S}$ such that $\wp'_0 \in [\wp''_0]$. Since $\wp'_i \in [\wp]$ for each $i \geq 1$ (hence, $\rightsquigarrow_{\wp'_i}$ implies $\rightsquigarrow_\wp$), it follows that for each $n \in \mathbb{N}$, there is a finite run π''_n starting from s which is an instance of the finite path $\wp''_0 \underbrace{\wp \ldots \wp}_{n \text{ times}}$. Hence, s is neatly unbounded w.r.t. the *ultimately periodic* path $\wp''_0(\wp)^\omega$. Moreover, $\wp''_0, \wp \in \mathcal{P}_\mathcal{S}$, $G_{\wp''_0} \cdot G_\wp$ is satisfiable, $G_\wp$ is idempotent, and by Theorem 1, $G_{\wp''_0}$ and $G_\wp$ are complete and normalized, which concludes. ☐

B.2 Detailed proof of Theorem 8

Theorem 8. *Let $\mathcal{S}$ be a* GCS. *Then, $Unb_\mathcal{S}$ is* MG *representable and one can construct a* MG *representation of $Unb_\mathcal{S}$. Moreover, given $q \in Q(\mathcal{S})$, checking whether $Unb^q_\mathcal{S} \neq \emptyset$ is in* PSPACE *and can be done in time $O(|E(\mathcal{S})| \cdot |Q(\mathcal{S})|^2 \cdot (K+2)^{(2|Var|+|Const|)^2})$.*

Proof. We have already proved Theorem 8 in case the given GCS $\mathcal{S}$ is assumed to be complete. Now, assume that $\mathcal{S}$ is not complete. Then, the following holds.

Claim: one can build a complete GCS $\mathcal{C}(\mathcal{S})$ having the same variables and constants as $\mathcal{S}$ s.t. $[\![\mathcal{C}(\mathcal{S})]\!] = [\![\mathcal{S}]\!]$, $Q(\mathcal{C}(\mathcal{S})) = Q(\mathcal{S})$, and $E(\mathcal{C}(\mathcal{S})) = O(E(\mathcal{S}) \cdot 2^{(2|Var|+|Const|)^2})$.

Proof of the claim: we need an additional definition. A *basic complete transitional* MG G is a transitional MG such that

- the weight of each edge is in $\{0, 1\}$;
- for all vertices u and v, either $G \models u \lhd v$ or $G \models v \lhd u$ for some $\lhd \in \{<, =\}$.

Let $\mathcal{G}_b$ be the set of basic complete transitional MG. Evidently, $\mathcal{G}_b$ is finite and its cardinality is bounded by $O(2^{(2|Var|+|Const|)^2})$. Moreover, note that for each transitional MG G and $G' \in \mathcal{G}_b$, $G \otimes G'$ is complete. Furthermore, $Sat(G) = \bigcup_{G' \in \mathcal{G}_b} Sat(G \otimes G')$. Then, $\mathcal{C}(\mathcal{S})$ is obtained from $\mathcal{S}$ by replacing each edge $q \xrightarrow{G} q'$ with the edges $q \xrightarrow{G \otimes G'} q'$, where $G' \in \mathcal{G}_b$. Evidently, $[\![\mathcal{C}(\mathcal{S})]\!] = [\![\mathcal{S}]\!]$, $Q(\mathcal{C}(\mathcal{S})) = Q(\mathcal{S})$, and $E(\mathcal{C}(\mathcal{S})) = O(E(\mathcal{S}) \cdot 2^{(2|Var|+|Const|)^2})$. ☐

Since $[\![\mathcal{C}(\mathcal{S})]\!] = [\![\mathcal{S}]\!]$, it holds that $Unb_\mathcal{S} = Unb_{\mathcal{C}(\mathcal{S})}$. Thus, since the theorem holds for complete GCS, by the claim above it follows that one can compute a MG representation of $Unb_\mathcal{S}$, and given $q \in Q(\mathcal{S})$, checking whether $Unb^q_\mathcal{S} \neq \emptyset$ can be done in time $O(|E(\mathcal{S})| \cdot |Q(\mathcal{S})|^2 \cdot (K+2)^{(2|Var|+|Const|)^2})$. Note that by the proof of the claim above, an edge of $\mathcal{C}(\mathcal{S})$ is of the form $q \xrightarrow{G \otimes G'} q'$, where $q \xrightarrow{G} q'$ is an edge of $\mathcal{S}$, and G' is an arbitrary basic complete transitional MG. Thus, by a trivial readaptation of the NPSPACE algorithm used to check whether $Unb^q_\mathcal{S} \neq \emptyset$ in case $\mathcal{S}$ is assumed to be complete (see proof of Theorem 8 in Section 4), the result follows. ☐

B.3 Membership in PSPACE for strong termination w.r.t. a designated state

The proof is by a linear-time reduction to the problem of checking for a given GCS $\mathcal{S}$ and control point $q \in Q(\mathcal{S})$, whether $Unb^q_\mathcal{S} = \emptyset$ (by Theorem 8 this last problem is in PSPACE). Fix a GCS $\mathcal{S}$ and a state (q_0, ν_0) of $\mathcal{S}$. W.l.o.g. we can assume that $\nu_0(x) \in Const$ for each $x \in Var$ (otherwise, we extend $Const$ by including the integers $\nu_0(x)$ with $x \in Var$). Let $G_=$ be the transitional MG corresponding to the GC given by $\bigwedge_{x \in Var} x = \nu_0(x)$, and $q_0' \notin Q(\mathcal{S})$ be a fresh control point. We construct a new GCS $\mathcal{S}_0$ as follows: $\mathcal{S}_0$ is obtained from $\mathcal{S}$ by adding for each edge of $\mathcal{S}$ of the form $q_0 \xrightarrow{G} q$, the edge $q_0' \xrightarrow{G \otimes G_=} q$. By construction it easily follows that $(q_0, \nu_0) \notin Unb_{\mathcal{S}}$ iff $Unb_{\mathcal{S}_0}^{q_0'} = \emptyset$. Hence, the result follows.

C Proof of Theorem 9

We need a preliminary result.

Proposition 6. *Given a GCS $\mathcal{S}$, a state s_0 of $\mathcal{S}$, and $q_0 \in Q(\mathcal{S})$, checking whether there is a maximal finite run from s_0 leading to a state of the form (q_0, ν) for some valuation ν is decidable.*

Proof. W.l.o.g. we can assume that $\mathcal{S}$ is complete. By Theorem 2 and Proposition 2, it easily follows that one can compute a MG representation $\{\mathcal{G}_q\}_{q \in Q(\mathcal{S})}$ of the set of states which are reachable from s_0 in $[\![\mathcal{S}]\!]$. Moreover, by Proposition 2, one can compute a MG representation $\{\mathcal{G}_q^1\}_{q \in Q(\mathcal{S})}$ of the set of states which have some successor in $[\![\mathcal{S}]\!]$. Thus, there is a maximal finite run from s_0 leading to a state of the form (q_0, ν) for some valuation ν *if and only if* there is $G \in \mathcal{G}_{q_0}$ and $\nu \in Sat(G)$ such that $\nu \notin Sat(G')$ for each $G' \in \mathcal{G}_{q_0}^1$. This last condition can be effectively checked since it can be effectively expressed in (quantifier-free) Presburger arithmetic.[5] Hence, the result follows. □

Now, we can prove the desired result.

Theorem 9. *The bounded universal eventually problem is decidable.*

Proof. Fix a GCS $\mathcal{S}$, a state s_0 of $\mathcal{S}$, and a control point $q \in Q(\mathcal{S})$. Let $\mathcal{S}_q$ be the GCS obtained from $\mathcal{S}$ by removing each edge starting from q. Evidently, the following holds: $(\mathcal{S}, s_0, q)$ is a positive instance of the bounded universal problem *if and only if* state s_0 is strongly-terminating in $\mathcal{S}_q$ (i.e., $s_0 \notin Unb_{\mathcal{S}_q}$) and there is *no* finite maximal run of $\mathcal{S}_q$ from s_0 leading to some state (q', ν) with $q' \neq q$. Thus, from Corollary 2 and Proposition 6, the result follows. □

[5] note that GC or, equivalently, MG correspond to a fragment of quantifier-free Presburger arithmetic.