Available online at www.sciencedirect.com
SCIENCE @ DIRECT®
Theoretical Computer Science 360 (2006) 352–372
Theoretical Computer Science
www.elsevier.com/locate/tcs

Model checking for process rewrite systems and a class of action-based regular properties

Laura Bozzelli*

Dipartimento di Matematica e Applicazioni, Università di Napoli "Federico II", Via Cintia, 80126 Napoli, Italy

Received 16 February 2005; received in revised form 23 January 2006; accepted 3 May 2006

Communicated by G. Levi

Abstract

We consider the model checking problem for Process Rewrite Systems (*PRS*), an infinite-state formalism (non Turing-powerful) which subsumes many common models such as Pushdown Processes and Petri Nets. *PRS* can be adopted as a formal model for programs with dynamic creation and synchronization of concurrent processes, and with recursive procedures. The model-checking problem of *PRS* against action-based linear temporal logic (*ALTL*) is undecidable. However, decidability for some interesting fragment of *ALTL* remains an open question. In this paper, we state decidability results concerning generalized acceptance properties about infinite derivations (infinite term rewriting) in *PRS*. As a consequence, we obtain decidability of the model-checking problem (restricted to infinite runs) of *PRS* against a meaningful fragment of *ALTL*.

Keywords: Infinite-state systems; Process rewrite systems; Petri nets; Pushdown processes; Model checking; Action-based linear temporal logic

1. Introduction

Automatic verification of systems is nowadays one of the most investigated topics. A major difficulty to face when considering this problem is that reasoning about systems in general may require dealing with infinite-state models. Software systems may introduce infinite states both manipulating data ranging over infinite domains, and having unbounded control structures such as recursive procedure calls and/or dynamic creation of concurrent processes (e.g. multi-threading). Many formalisms have been proposed for the description of infinite-state systems. Among the most popular are the well-known formalisms of Context Free Processes, Pushdown Processes, Petri Nets, and Process Algebras. The first two are models of sequential computation, whereas Petri Nets and Process Algebra explicitly take into account concurrency. The model checking problem for these infinite-state formalisms has been studied in the literature. As far as Context Free Processes and Pushdown Processes are concerned, decidability of the modal μ-calculus, the most powerful of the modal and temporal logics used for verification, has been established (see [1,4,9,13]). In [5,7,8], model checking for Petri nets has been studied. For branching temporal logics, the problem is undecidable even for restricted logics. Fortunately, model checking against action-based linear temporal logic (*ALTL*) [7,8] is decidable.

* Tel.: +39 081 662316; fax: +39 081 7145110.
E-mail address: laura.bozzelli@dma.unina.it.

0304-3975/$ - see front matter
doi:10.1016/j.tcs.2006.05.003

Verification of formalisms which accommodate both parallelism and recursion is a challenging problem. In order to formally study this kind of systems, recently the formal framework of Process Rewrite Systems (*PRS*) has been introduced [11,12]. This framework (non Turing-powerful), which is based on term rewriting, subsumes many common infinite-states models such as Pushdown Processes and Petri Nets. *PRS* can be adopted as a formal model for programs with dynamic creation and (a restricted form of) synchronization of concurrent processes, and with recursive procedures. The decidability results already known in the literature for the general framework of *PRS* concern the reachability problem between two fixed terms and the *reachable property* problem [11,12]. The latter is the problem of deciding whether there is a reachable term that satisfies certain properties that can be encoded as follows: some given rewrite rules are applicable and/or other given rewrite rules are *not* applicable. Decidability of this problem can be also used to decide the deadlock reachability problem. Recently, in [3], symbolic reachability analysis is investigated (i.e., the constructibility problem of the potentially infinite set of terms that are reachable from a given possibly infinite set of terms). However, the algorithm given in [3] can be applied only to a subclass of *PRS* (strictly less expressive), i.e., the *synchronization-free PRS* (the so-called PAD systems) which subsume Pushdown processes and the *synchronization-free* Petri nets. Concerning the *ALTL* model-checking problem, it is undecidable for the whole class of *PRS* [2,11,12]. It remains undecidable even for restricted models such as PA processes [2] (these systems correspond to a subclass, strictly less expressive, of PAD systems). However, Bouajjani et al. in [2] proved that for the complement of *simple ALTL* [1] (*simple ALTL* corresponds to Büchi ω-automata where there are only self-loops), model-checking PA processes is decidable. Anyway, decidability for some interesting fragment of *ALTL* and the general framework of *PRS* remains an open question.

In this paper we prove decidability of the model-checking problem (restricted to infinite runs) for the whole class of *PRS* w.r.t. a meaningful fragment of *ALTL* that captures, exactly, the class of regular properties invariant under permutation of atomic actions (along infinite runs). This fragment (closed under boolean connectives) is defined as follows:

$$\varphi ::= F\,\psi \mid GF\,\psi \mid \neg\varphi \mid \varphi \wedge \varphi, \tag{1}$$

where ψ is an *ALTL* propositional formula (i.e., a boolean combination of atomic actions). Within this fragment, classes of properties useful in system verification can be expressed: some *safety properties* (e.g., $G\,\psi_1$), some *guarantee properties* (e.g., $F\,\psi_1$), some *obligation properties* (e.g., $F\,\psi_1 \rightarrow F\,\psi_2$, or $G\,\psi_1 \rightarrow G\,\psi_2$), some *recurrence properties* (e.g., $GF\,\psi_1$), some *persistence properties* (e.g., $FG\,\psi_1$), and finally some *reactivity properties* (e.g., $GF\,\psi_1 \rightarrow GF\,\psi_2$). [2] Note that important classes of properties like invariants, as well as strong and weak fairness constraints, can be expressed. Moreover, note that this fragment and *simple ALTL* are incomparable (in particular, fairness cannot be expressed by *simple ALTL*).

In order to prove our result, we introduce the notion of *Multi Büchi Rewrite System* (*MBRS*) that is, informally speaking, a *PRS* augmented with a finite number of accepting components, where each component is a subset of the *PRS*. Then, we reduce our initial problem to that of verifying the existence of infinite derivations (infinite term rewriting) in *MBRS* satisfying *generalized* acceptance properties (*a la* Büchi). Finally, we prove decidability of this last problem by a reduction to the *ALTL* model-checking problem for Petri nets and Pushdown processes (that is known to be decidable). There are two main steps in the proof of decidability:

- First, we prove decidability of a problem concerning the existence of *finite* derivations leading to a given term and satisfying generalized acceptance properties. This problem is strictly more general than reachability problem and is not comparable with the reachable property problem of Mayr [11,12]. Moreover, our approach is substantially different from that used by Mayr.
- The second step concerns reasoning about infinite derivations in *PRS* which have not been investigated (to the best of our knowledge) in other papers on *PRS*.

Plan of the paper: In Section 2, we recall the framework of *PRS* and *ALTL* logic. In Section 3, we introduce the notion of *MBRS*, and show how our decidability result about generalized acceptance properties of infinite derivations in

---

[1] *Simple ALTL* is *not* closed under negation, and is defined as follows:

$$\varphi ::= \psi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \langle a\rangle\varphi \mid G\,\psi \mid \psi\,U\,\varphi,$$

where ψ is an *ALTL* propositional formula, a is an atomic action, and $\langle a\rangle$, G, and U are the *next*, *always*, and *until* operators.

[2] ψ_1 and ψ_2 denote *ALTL* propositional formulae.

PRS can be used in model-checking for the *ALTL* fragment (1). In Sections 4 and 5, we prove our decidability result. In Section 6, we examine the complexity of the considered problem. Finally, Appendix A contains some technical proofs.

Remark. We consider the model checking problem restricted to infinite runs since the only meaningful properties within the *ALTL* fragment (1) that can expressed about *finite* runs consist of positive boolean combinations of formulas either of the form $F\psi$ or of the form $G\psi$ (where ψ is an *ALTL* propositional formula). In the framework of *MBRS*, checking these properties for finite runs can be reduced to the problem about the existence of maximal finite derivations (i.e., finite derivations terminating in terms without successors) satisfying given (generalized) acceptance properties. This problem can be resolved by a simple modification of the algorithm presented in Subsection 4.1. Therefore, we expose only the results on infinite runs, the most interesting case with respect to the considered *ALTL* fragment.

2. Preliminaries

2.1. Process rewrite systems

Let $Var = \{X, Y, \ldots\}$ be a finite set of *process variables*. The set T of *process terms* t over Var is defined by the following syntax:

$$t ::= \varepsilon \mid X \mid t.t \mid t\|t,$$

where $X \in Var$, ε is the empty term, and "$\|$" (resp., ".") denotes parallel composition (resp., sequential composition). We always work with equivalences classes of process terms modulo commutativity and associativity of "$\|$", and modulo associativity of ".". Moreover, ε will act as the identity for both parallel and sequential composition, i.e., $\varepsilon.t = t.\varepsilon = t\|\varepsilon = t$.

Definition 1 (*Mayr [12]*). A *PRS* over *Var* and an alphabet of atomic actions Σ is a *finite* set of rewrite rules of the form $t \xrightarrow{a} t'$, where t ($\neq \varepsilon$) and t' are terms in T, and $a \in \Sigma$.

The set of process variables occurring in the rewrite rules of a *PRS* $\Re$ is denoted by $Var(\Re)$.

A *PRS* $\Re$ induces a Labelled Transition System (LTS) with set of states T, and a transition relation $\rightarrow \subseteq T \times \Sigma \times T$ defined by the following inference rules:

$$\frac{(t \xrightarrow{a} t') \in \Re}{t \xrightarrow{a} t'} \qquad \frac{t_1 \xrightarrow{a} t_1'}{t_1 \| t \xrightarrow{a} t_1' \| t} \qquad \frac{t_1 \xrightarrow{a} t_1'}{t_1.t \xrightarrow{a} t_1'.t},$$

where t, t', t_1, t_1' are process terms and $a \in \Sigma$. In a similar way we define for each rule $r \in \Re$, the notion of *single-step derivation* relation by r, denoted by $\stackrel{r}{\Rightarrow}_{\Re}$.

A *path in* $\Re$ from $t \in T$ is a (finite or infinite) sequence $\pi = t_0 \xrightarrow{a_0} t_1 \xrightarrow{a_1} t_2 \ldots$ such that $t = t_0$ and every triple $t_i \xrightarrow{a_i} t_{i+1}$ is a LTS edge. We write π^i for the path $t_i \xrightarrow{a_i} t_{i+1} \xrightarrow{a_{i+1}} \ldots$. Let $\mathit{firstact}(\pi) := a_0$. A *run in* $\Re$ is a maximal path, i.e., a path that is either infinite, or terminates in a term without successors. We denote by $runs(\Re)$ the set of all the runs in $\Re$.

A *derivation* in $\Re$ from $t \in T$ (through a sequence $\sigma = r_1 r_2 \ldots$ of rules in $\Re$) is a sequence d of the form $t_0 \stackrel{r_1}{\Rightarrow}_{\Re} t_1 \stackrel{r_2}{\Rightarrow}_{\Re} t_2 \ldots$ such that $t_0 = t$ and each triple $t_i \stackrel{r_i}{\Rightarrow}_{\Re} t_{i+1}$ is a single-step derivation. If d is finite and terminates in the term t', we say t' is *reachable* in $\Re$ from t (through derivation d). If σ is empty, we say d is a *null derivation*. For terms $t, t' \in T$ and a rule sequence σ, we write $t \stackrel{\sigma}{\Rightarrow}_{\Re}$ (resp., $t \stackrel{\sigma}{\Rightarrow}_{\Re} t'$), or just $t \stackrel{\sigma}{\Rightarrow}$ (resp., $t \stackrel{\sigma}{\Rightarrow} t'$) when $\Re$ is clear from the context, to mean that there is a derivation (resp., a finite derivation leading to t') from t through σ.

In the following, sometime, a rule sequence in $\Re$ is also seen as a mapping $\sigma : N' \rightarrow \Re$ where N' is a subset of $\mathbb{N}$. We denote by $|\sigma|$ the length of σ, by $pr(\sigma)$ the set N', and by $\min(N')$ the smallest element of N'. A rule sequence $\sigma' : N'' \rightarrow \Re$ is a subsequence of $\sigma : N' \rightarrow \Re$ iff $N'' \subseteq N'$ and $\sigma' = \sigma|_{N''}$, that is σ' is the restriction of σ to the set N''. Given two rule sequences σ and σ', we say that they are *disjoint* iff $pr(\sigma) \cap pr(\sigma') = \emptyset$. For a rule sequence σ in $\Re$, and a subsequence σ' of σ, $\sigma\backslash\sigma'$ denotes the rule sequence obtained by removing from σ all and only the occurrences of rules in σ'.

Now, we define *interleavings* of rule sequences in a *PRS*.

Definition 2. The *interleaving* of two rule sequences σ_1 and σ_2, denoted by $Interleave(\sigma_1, \sigma_2)$, is the set of rule sequences defined as follows (where o denotes the empty sequence):

$$Interleave(\sigma_1, o) := Interleave(o, \sigma_1) := \{\sigma_1\};$$

$$Interleave(r_1\sigma_1, r_2\sigma_2) := \{r_1\sigma \mid \sigma \in Interleave(\sigma_1, r_2\sigma_2)\} \cup \{r_2\sigma \mid \sigma \in Interleave(r_1\sigma_1, \sigma_2)\}$$

The generalization of the function *Interleave* to arbitrary sequences $(\sigma_h)_{h=0}^{h=m}$ (where $m \in \mathbb{N}\cup\{\infty\}$) of rule sequences in a *PRS* is straightforward.
The proof of the following Proposition is simple.

Proposition 1. *Let σ be a rule sequence in $\Re$ and $(\sigma_h)_{h=0}^{h=m}$ (where $m \in \mathbb{N} \cup \{\infty\}$) be a sequence of subsequences of σ two by two disjoints and such that $\bigcup_{h=0}^{h=m} pr(\sigma_h) = pr(\sigma)$. Then, $\sigma \in Interleave((\sigma_h)_{h=0}^{h=m})$.*

For technical reasons, we also consider *PRS* in a restricted syntactical form called *normal form* [12]. A *PRS* $\Re$ is said to be in *normal form* if every rule $r \in \Re$ has one of the following forms:

PAR rules: $X_1\|X_2\ldots\|X_p \xrightarrow{a} Y_1\|Y_2\ldots\|Y_q$ where $p \in \mathbb{N}\backslash\{0\}$ and $q \in \mathbb{N}$.

SEQ rules: $X \xrightarrow{a} Y.Z$ or $X.Y \xrightarrow{a} Z$ or $X \xrightarrow{a} Y$ or $X \xrightarrow{a} \varepsilon$

with $X, Y, Z, X_i, Y_j \in Var$. A *PRS* where all the rules are SEQ (resp., PAR) rules is called *sequential* (resp., *parallel*) *PRS*.

2.2. ALTL (Action-based LTL) and PRS

The set of *ALTL* formulae over a set Σ of atomic actions is defined as follows:

$$\varphi ::= true \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle a\rangle\varphi \mid \varphi U \varphi ,$$

where $a \in \Sigma$, $\langle a\rangle$ is the relativised *next* operator (one for each action $a \in \Sigma$), and U is the *until* operator. We also define $F\varphi := true\, U\, \varphi$ ("*eventually* φ") and its dual $G\varphi := \neg F\neg\varphi$ ("*always* φ"). Given an *PRS* $\Re$ and an *ALTL* formula φ, the set of the runs in $\Re$ *satisfying* φ, in symbols $[[\varphi]]_\Re$, is defined inductively on the structure of φ as follows:

- $[[true]]_\Re = runs(\Re)$,
- $[[\neg\varphi]]_\Re = runs(\Re)\backslash[[\varphi]]_\Re$,
- $[[\varphi_1 \wedge \varphi_2]]_\Re = [[\varphi_1]]_\Re \cap [[\varphi_2]]_\Re$,
- $[[\langle a\rangle\varphi]]_\Re = \{\pi \in runs(\Re) \mid firstact(\pi) = a \text{ and } \pi^1 \in [[\varphi]]_\Re\}$,
- $[[\varphi_1 \mathcal{U} \varphi_2]]_\Re = \{\pi \in runs(\Re) \mid \exists\, i\,.\, \pi^i \in [[\varphi_2]]_\Re \text{ and } \forall\, j < i.\, \pi^j \in [[\varphi_1]]_\Re\}$.

The *ALTL* model-checking problem (resp., *ALTL* model-checking problem restricted to infinite runs) of *PRS* is the problem of deciding whether, for a *PRS* $\Re$, an *ALTL* formula φ and a term $t \in T$, all the runs (resp., infinite runs) of $\Re$ starting at t belong to $[[\varphi]]_\Re$. The following is a well-known result:

Proposition 2 (*see [1,7,11]*). *The* ALTL *model-checking problem of parallel* (*resp*., *sequential*) *PRS*, *possibly restricted to infinite runs*, *is decidable*.

In this paper we are interested in the model-checking problem (restricted to infinite runs) of unrestricted *PRS* against the following *ALTL* fragment:

$$\varphi ::= F\,\psi \mid GF\,\psi \mid \neg\varphi \mid \varphi \wedge \varphi, \tag{2}$$

where ψ denotes an *ALTL propositional* formula defined by the following syntax: $\psi ::= \langle a\rangle\ true \mid \psi \wedge \psi \mid \neg\psi$ (where $a \in \Sigma$). We denote the *ALTL* fragment (2) by $\mathcal{F}$.

3. Multi Büchi rewrite systems

In order to solve the model-checking problem (restricted to infinite runs) of *PRS* against the *ALTL* fragment $\mathcal{F}$, we encode it in a suitable framework. Therefore, we introduce the notion of *MBRS* that is a *PRS* augmented with a tuple of subsets of the given *PRS* called *accepting components*.

Definition 3. A *MBRS* (with *n accepting components*) over *Var* and Σ is a tuple $M = \langle \Re, \langle \Re_1, \ldots, \Re_n \rangle\rangle$, where $\Re$ is a *PRS* over *Var* and Σ, and for all $i = 1, \ldots, n$, $\Re_i \subseteq \Re$. $\Re$ is called the *support* of *M*.

We say that *M* is an *MBRS in normal form* (resp., *sequential MBRS*, *parallel MBRS*) if the support $\Re$ is in normal form (resp., sequential, parallel).

For a rule sequence σ in $\Re$, the *finite acceptance of* σ w.r.t. *M*, denoted by $\Upsilon_M^f(\sigma)$, is the set $\{i \in \{1, \ldots, n\} | \ \sigma$ contains some occurrence of rule in $\Re_i\}$. The *infinite acceptance of* σ w.r.t. *M*, denoted by $\Upsilon_M^\infty(\sigma)$, is the set $\{i \in \{1, \ldots, n\} | \ \sigma$ contains infinite occurrences of some rule in $\Re_i\}$. Given $K, K_\omega \subseteq \{1, \ldots, n\}$ and a derivation *d* of the form $t \overset{\sigma}{\Rightarrow}_\Re$, we say that *d* is a (K, K_ω)-*accepting derivation* in *M* if $\Upsilon_M^f(\sigma) = K$ and $\Upsilon_M^\infty(\sigma) = K_\omega$. Moreover, we say that *d* has *finite acceptance* (resp., *infinite acceptance*) *K* (resp., K_ω) in *M*. We denote by P_n the set $2^{\{1,\ldots,n\}}$ (i.e., the set of the subsets of $\{1, \ldots, n\}$). For any $K \in P_n$, we denote by $|K|$ the cardinality of *K*.

Let $(K_h)_{h=0}^{h=m}$ be a sequence of elements in P_n (where $m \in \mathbb{N} \cup \{\infty\}$). We denote by $\bigoplus_{h=0}^{h=m} K_h$ the element of P_n given by $\{i |$ for infinitely many $h \in \mathbb{N}, i \in K_h\}$. Obviously, if *m* is finite, then $\bigoplus_{h=0}^{h=m} K_h$ is empty. The proof of the following Proposition is simple.

Proposition 3. *Let* $(\sigma_h)_{h=0}^{h=m}$ *be a sequence of rule sequences in* $\Re$ (*where* $m \in \mathbb{N} \cup \{\infty\}$). *Then, for all* $\sigma \in \mathit{Interleaving}((\sigma_h)_{h=0}^{h=m})$ *we have*
(1) $\Upsilon_M^f(\sigma) = \bigcup_{h=0}^{h=m} \Upsilon_M^f(\sigma_h)$.
(2) $\Upsilon_M^\infty(\sigma) = \bigcup_{h=0}^{h=m} \Upsilon_M^\infty(\sigma_h) \cup \bigoplus_{h=0}^{h=m} \Upsilon_M^f(\sigma_h)$.

Now, let us consider the following problem:

Fairness Problem. *Given an* MBRS $M = \langle \Re, \langle \Re_1, \ldots, \Re_n \rangle\rangle$ *over Var and* Σ, *a process term t, and two sets* $K, K_\omega \in P_n$, *is there a* (K, K_ω)-*accepting infinite derivation in M from t*?

Without loss of generality we can assume that the input term *t* in the Fairness Problem is a process variable. In fact, if $t \notin \mathit{Var}$, then we add a fresh variable *X* and a rule of the form $X \to t$ whose finite acceptance is the empty set.

Theorem 1. *Model-checking* PRS (*restricted to infinite runs*) *against the* ALTL *fragment* $\mathcal{F}$ *is polynomial-time reducible to the Fairness Problem.*

Proof. We fix a *PRS* $\Re$ over *Var* and Σ. For an *ALTL* propositional formula ψ over Σ, we denote by $[[\psi]]_\Sigma$ the subset of Σ inductively defined as follows: (1) for all $a \in \Sigma$, $[[\langle a \rangle \mathit{true}]]_\Sigma = \{a\}$; (2) $[[\neg\psi]]_\Sigma = \Sigma \setminus [[\psi]]_\Sigma$; (3) $[[\psi_1 \wedge \psi_2]]_\Sigma = [[\psi_1]]_\Sigma \cap [[\psi_2]]_\Sigma$. Evidently, given an infinite run π of $\Re$, we have that $\pi \in [[\psi]]_\Re$ iff $\mathit{firstact}(\pi) \in [[\psi]]_\Sigma$. For a rule $r = t \overset{a}{\to} t' \in \Re$, we say that *r satisfies* ψ iff $a \in [[\psi]]_\Sigma$. We denote by $AC_\Re(\psi)$ the set of rules in $\Re$ that satisfy ψ.

Now, we prove the assertion. Given a term *t* and a formula φ of the *ALTL* fragment $\mathcal{F}$, we have to decide whether all the infinite runs of $\Re$ starting at *t* satisfy φ or, equivalently, whether there is an infinite run starting at *t* satisfying $\neg\varphi$. Let us consider the derivative operator $F^+\varphi_1 := F\varphi_1 \wedge \neg GF\varphi_1$. By using the following logic equivalences:
• $G\varphi_1 \wedge G\varphi_2 \equiv G(\varphi_1 \wedge \varphi_2)$, $\neg F\varphi_1 \equiv G\neg\varphi_1$, $\neg G\varphi_1 \equiv F\neg\varphi_1$, $F\varphi_1 \equiv F^+\varphi_1 \vee GF\varphi_1$, $FG\varphi_1 \equiv F^+\neg\varphi_1 \vee G\varphi_1$,
formula $\neg\varphi$ can be rewritten in the following disjunctive normal form

$$\neg\varphi \equiv \bigvee \left(\bigwedge_j F^+\psi_j \wedge \bigwedge_k GF\eta_k \wedge G\zeta \right), \tag{3}$$

where ψ_j, η_k, and ζ are *ALTL* propositional formulae. Evidently, we can restrict ourselves to consider a single disjunct in (3), i.e., a formula having the form

$$F^+\psi_1 \wedge \cdots \wedge F^+\psi_{m_1} \wedge GF\eta_1 \wedge \cdots \wedge GF\eta_{m_2} \wedge G\zeta. \tag{4}$$

Let us consider the *MBRS* $M = \langle \overline{\Re}, \langle \Re_1, \ldots, \Re_n \rangle \rangle$ where $\overline{\Re} = AC_{\Re}(\zeta)$, $n = m_1 + m_2$, and

$$\text{for all } i = 1, \ldots, m_1, \quad \Re_i = AC_{\overline{\Re}}(\psi_i),$$
$$\text{for all } j = 1, \ldots, m_2, \quad \Re_{j+m_1} = AC_{\overline{\Re}}(\eta_j).$$

Let $K = \{1, \ldots, n\}$ and $K_\omega = \{m_1 + 1, \ldots, n\}$. It is easy to deduce that there is an infinite run starting at t satisfying formula (4) *iff* there is a (K, K_ω)-accepting infinite derivation in *M* from t. Since the reduction described above is computable in polynomial time, the thesis holds. □

In the remainder of this paper we prove that the Fairness Problem is decidable. We proceed in two main steps. First, in Section 4 we decide the problem for the class of *MBRS* in normal form. Then, in Section 5 we extend the result to the whole class of *MBRS*. For the proof we need some preliminary decidability results, stated by the following Propositions 4–6, concerning the existence of derivations in parallel and sequential *MBRS* satisfying given acceptance properties.

Proposition 4. *Given a* parallel MBRS $M_P = \langle \Re_P, \langle \Re_{P,1}, \ldots, \Re_{P,n} \rangle \rangle$ *over Var*, $X, Y \in Var$, *and* $K, K_\omega \in P_n$, *it is decidable whether*:

1. *there is a derivation of the form* $X \stackrel{\sigma}{\Rightarrow} t$ *such that* $t \neq \varepsilon$ *and* $\Upsilon^f_{M_P}(\sigma) = K$;
2. *there is a derivation of the form* $X \stackrel{\sigma}{\Rightarrow} \varepsilon$ *such that* $\Upsilon^f_{M_P}(\sigma) = K$;
3. *there is a derivation of the form* $X \stackrel{\sigma}{\Rightarrow} Y$ *such that* $\Upsilon^f_{M_P}(\sigma) = K$;
4. *there is a derivation of the form* $X \stackrel{\sigma}{\Rightarrow} t \| Y$ *such that* $|\sigma| > 0$ *and* $\Upsilon^f_{M_P}(\sigma) = K$;
5. *there is a* (K, K_ω)*-accepting infinite derivation in* M_P *from X*;

Proof. We prove decidability of conditions 1–5 by a polynomial-time reduction to the *ALTL* model-checking problem for *parallel PRS*, which is decidable (see Proposition 2). In particular, we build a new parallel *PRS* $\overline{\Re}_P$ and for each $i = 1, \ldots, 5$, an *ALTL* formula φ_i such that condition *i* holds *iff* there is a run in $\overline{\Re}_P$ starting at *X* that satisfies φ_i (i.e., the model-checking problem with input the parallel *PRS* $\overline{\Re}_P$, the initial term *X*, and formula $\neg\varphi_i$ has a negative answer).

Let $\Lambda = \Re_P \setminus \bigcup_{i \in \{1,\ldots,n\} \setminus K} \Re_{P,i}$. W.l.o.g. we can assume that Λ is not empty and $X, Y \in Var(\Lambda)$ (otherwise, checking conditions 1–4 is trivial). Moreover, we can assume that $K \supseteq K_\omega$ (otherwise, checking condition 5 is trivial). The parallel *PRS* $\overline{\Re}_P$ is defined over the alphabet of actions $\{\bot\} \cup Var(\Lambda) \cup \Lambda$, and it is obtained from Λ as follows. First, we substitute every rule *r* in Λ of the form $t \stackrel{a}{\rightarrow} t'$ with the rule $t \stackrel{r}{\rightarrow} t'$. Then, we add the rule $Y \stackrel{\bot}{\rightarrow} \varepsilon$, and for all $Z \in Var(\Lambda)$, the rule $Z \stackrel{Z}{\rightarrow} Z$. Note that, by construction, a term *t* over $Var(\Lambda)$ has no successor in $\overline{\Re}_P$ iff $t = \varepsilon$.

Let ψ, ψ_{acc}, and ψ^∞_{acc} be the following *ALTL* formulas,

$$\psi := \left(\bigvee_{Z \in Var(\Lambda)} \langle Z \rangle\, true \right) \vee (\langle \bot \rangle\, true) \vee \bigvee_{r \in \Lambda} \langle r \rangle\, true,$$

$$\psi_{\text{acc}} := \bigwedge_{j \in K} F \bigvee_{r \in \Re_{P,j} \cap \Lambda} \langle r \rangle\, true,^{3}$$

$$\psi^\infty_{\text{acc}} := \left(\bigwedge_{j \in K_\omega} GF \bigvee_{r \in \Re_{P,j} \cap \Lambda} \langle r \rangle\, true \right) \wedge \left(\bigwedge_{j \in K \setminus K_\omega} \neg GF \bigvee_{r \in \Re_{P,j} \cap \Lambda} \langle r \rangle\, true \right).$$

[3] If $\Re_{P,j} \cap \Lambda = \emptyset$, then $\bigvee_{r \in \Re_{P,j}} \langle r \rangle\, true$ denotes *false*.

Note that ψ_{acc} encodes the requirement that the finite acceptance of the derivation is K, while $\psi_{\text{acc}}^{\infty}$ requires that the infinite acceptance is K_ω.

The formulas $\varphi_1, \varphi_2, \varphi_3, \varphi_4$, and φ_5 that encode conditions 1, 2, 3, 4, and 5, respectively, are defined as follows:

$$\varphi_1 := \psi_{\text{acc}} \wedge \left(\left(\bigvee_{r \in \Lambda} \langle r \rangle \, true \right) U \bigvee_{Z \in Var(\Lambda)} G(\langle Z \rangle \, true) \right),$$

$$\varphi_2 := \psi_{\text{acc}} \wedge \left(\left(\bigvee_{r \in \Lambda} \langle r \rangle \, true \right) U \neg \varphi_1 \right),$$

$$\varphi_3 := \psi_{\text{acc}} \wedge \left(\left(\bigvee_{r \in \Lambda} \langle r \rangle \, true \right) U \langle \bot \rangle \neg \varphi_1 \right),$$

$$\varphi_4 := \psi_{\text{acc}} \wedge \left(\left(\bigvee_{r \in \Lambda} \langle r \rangle \, true \right) U \, G(\langle Y \rangle \, true) \right) \wedge F \bigvee_{r \in \Lambda} \langle r \rangle \, true,$$

$$\varphi_5 := \psi_{\text{acc}} \wedge \psi_{\text{acc}}^{\infty} \wedge G \bigvee_{r \in \Lambda} \langle r \rangle \, true. \qquad \square$$

Proposition 5. *Let M_{P_1} and M_{P_2} be two parallel* MBRS *with the same support $\Re_P$ and with n accepting components. Given $X \in Var$, $K, K_\omega \in P_n$, and a subset Λ of $\Re_P$, it is decidable whether*:

1. *there exists a derivation in $\Re_P$ of the form $X \overset{\sigma}{\Rightarrow}$ such that $\Upsilon_{M_{P_1}}^{f}(\sigma) = K$, $\Upsilon_{M_{P_1}}^{\infty}(\sigma) \cup \Upsilon_{M_{P_2}}^{f}(\sigma) = K_\omega$, and σ is either infinite or contains some occurrence of rule in Λ.*

Proof. Let $M_{P_1} = \langle \Re_P, \langle \Re_{1,1}, \ldots, \Re_{1,n} \rangle \rangle$ and $M_{P_2} = \langle \Re_P, \langle \Re_{2,1}, \ldots, \Re_{2,n} \rangle \rangle$. Moreover, let $M_P = \langle \Re_P, \langle \Re_1, \ldots, \Re_{2n+1} \rangle \rangle$ be the parallel *MBRS* defined as follows: for all $1 \leqslant i \leqslant n$, $\Re_i := \Re_{1,i}$ and $\Re_{i+n} := \Re_{2,i}$, and $\Re_{2n+1} = \Lambda$. Also, for any $K' \in P_n$, we define $K' + n := \{ j + n \mid j \in K' \}$.

Evidently, condition 1 holds iff one of the following two conditions holds:

- there is a finite derivation in M_P of the form $X \overset{\sigma}{\Rightarrow}$ such that $\Upsilon_{M_P}^{f}(\sigma) = K \cup (K_\omega + n) \cup \{2n+1\}$;
- there are $\overline{K}, \overline{K}_\omega \subseteq K_\omega$ such that $\overline{K} \cup \overline{K}_\omega = K_\omega$ and there is an infinite derivation in M_P of the form $X \overset{\sigma}{\Rightarrow}$ such that $\Upsilon_{M_P}^{f}(\sigma) = K \cup (\overline{K} + n)$ and $\overline{K}_\omega \subseteq \Upsilon_{M_P}^{\infty}(\sigma) \subseteq \overline{K}_\omega \cup \{n+1, \ldots, 2n\}$.

By Proposition 4 checking these two conditions is decidable. $\square$

Now, we give the notion of *s*-reachability in sequential *PRS*.

Definition 4. Given a *sequential PRS* $\Re_S$ over *Var*, and $X, Y \in Var$, Y is *s-reachable from X in* $\Re_S$ iff there exists a term t of the form $Y.X_1.X_2.\ldots.X_n$ (where $X_i \in Var$ for any $i = 1, \ldots, n$, and $n \geqslant 0$) such that $X \Rightarrow t$.

Proposition 6. *Given a sequential* MBRS $M_S = \langle \Re_S, \langle \Re_{S,1}, \ldots, \Re_{S,n} \rangle \rangle$ *over Var, two variables $X, Y \in Var$, and two sets $K, K_\omega \in P_n$, it is decidable whether*: (1) *Y is s-reachable from X in M_S through a $(K, \emptyset)$-accepting derivation*; (2) *there is a (K, K_ω)-accepting infinite derivation in M_S from X.*

Proof. We prove decidability of conditions 1 and 2 by a polynomial-time reduction to the *ALTL* model-checking problem for *sequential PRS*, which is decidable (see Proposition 2). In particular, we build a new sequential *PRS* $\overline{\Re}_S$ and for each $i = 1, 2$, an *ALTL* formula φ_i such that condition *i* holds *iff* there is a run in $\overline{\Re}_S$ starting at X that satisfies φ_i (i.e., the model-checking problem with input the sequential *PRS* $\overline{\Re}_P$, the initial term X, and formula $\neg\varphi_i$ has a negative answer).

Let $\Lambda = \Re_S \setminus \bigcup_{i \in \{1,\ldots,n\} \setminus K} \Re_{S,i}$. W.l.o.g. we can assume that Λ is not empty and $X, Y \in Var(\Lambda)$ (otherwise, checking conditions 1 and 2 is trivial). The sequential *PRS* $\overline{\Re}_S$ is defined over the alphabet of actions $Var(\Lambda) \cup \{Y\}$, and it is obtained from Λ as follows. First, we substitute every rule r in Λ of the form $t \stackrel{a}{\rightarrow} t'$ with the rule $t \stackrel{r}{\rightarrow} t'$. Then, we add the rule $Y \stackrel{Y}{\rightarrow} Y$.

Let ψ_{acc} and $\psi^{\infty}_{\mathrm{acc}}$ be the following *ALTL* formulas:

$$\psi_{\mathrm{acc}} := \bigwedge_{j \in K} F \bigvee_{r \in \Re_{S,j} \cap \Lambda} \langle r \rangle\, true,^{4}$$

$$\psi^{\infty}_{\mathrm{acc}} := \left(\bigwedge_{j \in K_\omega} GF \bigvee_{r \in \Re_{S,j} \cap \Lambda} \langle r \rangle\, true \right) \wedge \left(\bigwedge_{j \in K \setminus K_\omega} \neg GF \bigvee_{r \in \Re_{S,j} \cap \Lambda} \langle r \rangle\, true \right).$$

Note that ψ_{acc} encodes the requirement that the finite acceptance of the derivation is K, while $\psi^{\infty}_{\mathrm{acc}}$ requires that the infinite acceptance is K_ω. The formulas φ_1 and φ_2 that encode conditions 1 and 2, respectively, are defined as follows:

$$\varphi_1 := \psi_{\mathrm{acc}} \wedge FG(\langle Y \rangle\, true),$$

$$\varphi_2 := \psi_{\mathrm{acc}} \wedge \psi^{\infty}_{\mathrm{acc}} \wedge G \bigvee_{r \in \Lambda} \langle r \rangle\, true. \qquad \square$$

4. Decidability of the fairness problem for *MBRS* in normal form

In this section we prove decidability of the Fairness Problem for the class of *MBRS* in normal form. We fix an *MBRS* in normal form $M = \langle \Re, \langle \Re_1, \ldots, \Re_n \rangle \rangle$ over *Var* and Σ. Given $X \in Var$ and $(K, K_\omega) \in P_n \times P_n$, we have to decide whether there exists a (K, K_ω)-accepting infinite derivation in M from X.

Remark 1. Since M is in normal form (and in the following we only consider derivations starting at variables or terms in which no sequential composition occurs) we can limit ourselves to consider only *terms in normal form*, defined as $t ::= \varepsilon | X | t \| t | t.X$ (where $X \in Var$). In fact, given a term in normal form t, each term t' reachable from t in M is still in normal form.

Convention 1. We assume without loss of generality that *Var* is the finite set of all and only the variables occurring in the rewrite rules in $\Re$ (i.e., $Var(\Re) = Var$).

There are two main steps in the decidability proof of the Fairness Problem.

Step 1: First, we prove decidability of the following problem:

Finite acceptance problem: Given $X, Y \in Var$ and $K \in P_n$, is there a finite derivation in M of the form $X \stackrel{\sigma}{\Rightarrow}$ (resp., $X \stackrel{\sigma}{\Rightarrow} Y$) such that $\Upsilon^f_M(\sigma) = K$?

Step 2: Using decidability of the Finite Acceptance Problem, we show that the Fairness Problem can be reduced to (a combination of) simpler and decidable problems concerning the existence of derivations in parallel and sequential *MBRS* satisfying given acceptance properties.

Before illustrating our approach, we need additional notation.

In the following, $M_P = \langle \Re_P, \langle \Re_{P,1}, \ldots, \Re_{P,n} \rangle \rangle$ denotes the restriction of M to the PAR rules, i.e., $\Re_P$ (resp., $\Re_{P,i}$ for $i = 1, \ldots, n$) is the set of all and only the PAR rules of $\Re$ (resp., $\Re_i$ for $i = 1, \ldots, n$). Moreover, we use a fresh variable $Z_F \notin Var$, and denote by T (resp., T_{PAR}, T_{SEQ}) the set of process terms in normal form (resp., in which no sequential composition occurs, in which no parallel composition occurs) over $Var \cup \{Z_F\}$.

The following definition introduces the notion of *subderivation*.

Definition 5 (*Subderivation*). *Let* $\overline{t} \stackrel{\lambda}{\Rightarrow} t \| (s.X) \stackrel{\sigma}{\Rightarrow}$ *be a derivation*[5] *in* $\Re$. *The set of the* subderivations d' of $d = (t \| (s.X) \stackrel{\sigma}{\Rightarrow})$ from s *is inductively defined as follows*:

---

[4] If $\Re_{S,j} \cap \Lambda = \emptyset$, then $\bigvee_{r \in \Re_{S,j}} \langle r \rangle\, true$ denotes *false*.

[5] In the following, locations of the kind 'the derivation $t \stackrel{\sigma}{\Rightarrow}$, mean that (there is a derivation of this form) and we are considering a specific derivation of the form $t \stackrel{\sigma}{\Rightarrow}$, and $t \stackrel{\sigma}{\Rightarrow}$ is used as a reference to this derivation.

1. *if d is a null derivation* or $s = \varepsilon$ or *d is of the form $t \,\|(Z.X) \overset{r}{\Rightarrow} t\|Y \overset{\sigma'}{\Rightarrow}$ (with $r = Z.X \overset{a}{\rightarrow} Y$ and $s = Z$), then d' is the null derivation from s*;
2. *if d is of the form $t \,\|(s.X) \overset{r}{\Rightarrow} t\|(s'.X) \overset{\sigma'}{\Rightarrow}$ (with $s \overset{r}{\Rightarrow} s'$) and $s' \overset{\mu'}{\Rightarrow}$ is a subderivation of $t \,\|(s'.X) \overset{\sigma'}{\Rightarrow}$ from s', then $s \overset{r}{\Rightarrow} s' \overset{\mu'}{\Rightarrow}$ is a subderivation of d from s*;
3. *if d is of the form $t\|(s.X) \overset{r}{\Rightarrow} t'\|(s.X) \overset{\sigma'}{\Rightarrow}$ (with $t \overset{r}{\Rightarrow} t'$), then each subderivation of $t'\|(s.X) \overset{\sigma'}{\Rightarrow}$ from s is also a subderivation of d from s.*

Moreover, we say that d' is a subderivation of $\overline{t} \overset{\lambda}{\Rightarrow} t\|(s.X) \overset{\sigma}{\Rightarrow}$.

4.1. Step 1 (Finite acceptance problem)

We prove decidability of the finite acceptance problem by a reduction to a similar problem restricted to a parallel *MBRS* (that is decidable by Proposition 4). The main idea is to mimic finite derivations d in M of the form $p \overset{\sigma}{\Rightarrow} t$ such that $p \in T_{\text{PAR}}$ (preserving the initial term p, the finite acceptance of σ in M, and the final term t if $t \in T_{\text{PAR}}$) by using only PAR rules belonging to an extension, denoted by M_{PAR} (and with support $\Re_{\text{PAR}}$), of the parallel *MBRS* M_P (the restriction of M to the PAR rules). For the given derivation d, we individuate subderivations which intuitively correspond to "local" maximal computations starting at process variable Z and activated by "procedure calls", i.e., by the application of SEQ rules r of the form $X \overset{a}{\rightarrow} Z.Y$. Then, the idea is to introduce new PAR rules in order to keep track of the meaningful information of such a subderivation together with the "caller" rule r and possibly with the SEQ rule (of the form $W.Y \overset{b}{\rightarrow} W'$) corresponding to the "return" (if any) of this subderivation. In order to illustrate this, let us denote by $N_{\text{SEQ}}(\sigma)$ the number of occurrences in σ of SEQ rules of the form $X \overset{a}{\rightarrow} Z.Y$. We proceed by induction on $N_{\text{SEQ}}(\sigma)$. If $N_{\text{SEQ}}(\sigma) = 0$, then since $p \in T_{\text{PAR}}$, we deduce that $p \overset{\sigma}{\Rightarrow} t$ is also a derivation in M_P (and thus in M_{PAR}, since M_{PAR} is an extension of M_P). Now, let us assume that $N_{\text{SEQ}}(\sigma) > 0$. In this case $p \overset{\sigma}{\Rightarrow} t$ can be rewritten in the form $p \overset{\lambda}{\Rightarrow} \overline{p}\|X \overset{r}{\Rightarrow} \overline{p}\|(Z.Y) \overset{\nu}{\Rightarrow} t$ where $r = X \overset{a}{\rightarrow} Z.Y$, λ contains only occurrences of PAR rules in $\Re$, $\overline{p} \in T_{\text{PAR}}$ and $X, Y, Z \in Var$. Let $d' = (Z \overset{\rho}{\Rightarrow} t_1)$ be a subderivation of $\overline{p}\|(Z.Y) \overset{\nu}{\Rightarrow} t$ from Z. By the definition of subderivation only one of the following three cases may occur:

A $t_1 \neq \varepsilon$, $\overline{p} \overset{\nu\backslash\rho}{\Rightarrow} t_2$, and $t = t_2\|(t_1.Y)$. In this case the subderivation d' does *not* influence the applicability of rules in the context that is in parallel with the "caller" process Y.

B $t_1 = \varepsilon$ and $p \overset{\sigma}{\Rightarrow} t$ is of the form $p \overset{\lambda}{\Rightarrow} \overline{p}\|X \overset{r}{\Rightarrow} \overline{p}\|(Z.Y) \overset{\nu_1}{\Rightarrow} t_2\|Y \overset{\nu_2}{\Rightarrow} t$, where ρ is a subsequence of ν_1 and $\overline{p} \overset{\nu_1\backslash\rho}{\Rightarrow} t_2$. In this case d' corresponds to the computation of a procedure which terminates without modifying the "caller" process Y.

C $t_1 = W \in Var$, and $p \overset{\sigma}{\Rightarrow}$ can be written as

$$p \overset{\lambda}{\Rightarrow} \overline{p}\|X \overset{r}{\Rightarrow} \overline{p}\|(Z.Y) \overset{\nu_1}{\Rightarrow} t_2\|(W.Y) \overset{r'}{\Rightarrow} t_2\|W' \overset{\nu_2}{\Rightarrow} t, \tag{5}$$

where $r' = W.Y \overset{b}{\rightarrow} W'$, ρ is a subsequence of ν_1 and $\overline{p} \overset{\nu_1\backslash\rho}{\Rightarrow} t_2$.

In this case d' corresponds to the computation of a procedure which terminates and modifies the "caller" process Y (which becomes the process W').

Cases **A**, **B** and **C** can be dealt in a similar way, so that we examine only case **C**. By anticipating the application of the rules in $\rho r'$ before the application of the rules in $\nu_1\backslash\rho$ we obtain the following derivation that has the same finite acceptance as (5): $p \overset{\lambda}{\Rightarrow} \overline{p}\|X \overset{r}{\Rightarrow} \overline{p}\|(Z.Y) \overset{\rho}{\Rightarrow} \overline{p}\|(W.Y) \overset{r'}{\Rightarrow} \overline{p}\|W' \overset{\gamma}{\Rightarrow} t$, where $\gamma = (\nu_1\backslash\rho)\nu_2$. Since $Z \overset{\rho}{\Rightarrow} W$ with $Z, W \in Var$ and $N_{\text{SEQ}}(\rho) < N_{\text{SEQ}}(\sigma)$, by the induction hypothesis there will be a derivation in M_{PAR} having the form $Z \overset{\pi}{\Rightarrow} W$ with $\Upsilon^f_{M_{\text{PAR}}}(\pi) = \Upsilon^f_M(\rho)$. By Proposition 4 for each $K \in P_n$, it is decidable whether there exists in M_{PAR} a finite derivation starting from variable Z and leading to variable W, and having finite acceptance K (in M_{PAR}). Then, the idea is to collapse the finite derivation d'' given by $X \overset{r}{\Rightarrow} Z.Y \overset{\rho}{\Rightarrow} W.Y \overset{r'}{\Rightarrow} W'$ into a single PAR rule of the form $r'' = X \overset{K}{\rightarrow} W'$ such that $K = \Upsilon^f_M(rr') \cup \Upsilon^f_{M_{\text{PAR}}}(\pi) = \Upsilon^f_M(rr'\rho)$ and $\Upsilon^f_{M_{\text{PAR}}}(r'') = K$. Thus, rule r'' keeps track of the meaningful information of the derivation d'', i.e., the starting term $X \in Var$, the final term $W' \in Var$, and the finite acceptance of $rr'\rho$ in M. Since the set of rules of the form $X \stackrel{K}{\to} W'$ with $X, W' \in Var$ and $K \in P_n$ is finite, M_{PAR} can be built effectively. After all, we have that $p \stackrel{\lambda r''}{\Rightarrow}_{\Re_{\mathrm{PAR}}} \overline{p} \| W'$ and $\overline{p} \| W' \stackrel{\gamma}{\Rightarrow}_{\Re} t$ such that $\overline{p} \| W' \in T_{\mathrm{PAR}}$, $\Upsilon^f_{M_{\mathrm{PAR}}}(\lambda r'') = \Upsilon^f_M(\lambda r r' \rho)$ and $N_{\mathrm{SEQ}}(\gamma) < N_{\mathrm{SEQ}}(\sigma)$. Applying again the induction hypothesis, we deduce that there exists a finite derivation in M_{PAR} of the form $p \stackrel{\xi}{\Rightarrow} p'$ such that $\Upsilon^f_{M_{\mathrm{PAR}}}(\xi) = \Upsilon^f_M(\sigma)$, and $p' = t$ if $t \in T_{\mathrm{PAR}}$. The fresh variable Z_F is used to manage case **A**, where the subderivation $Z \stackrel{\rho}{\Rightarrow} t_1$ does not influence the applicability of rules in $\nu \backslash \rho$ (i.e., $\overline{p} \stackrel{\nu \backslash \rho}{\Rightarrow} t_2$). In this case, in order to keep track of the derivation $X \stackrel{r}{\Rightarrow} Z.Y \stackrel{\rho}{\Rightarrow} t_1.Y$, it is sufficient to preserve the starting term X and the finite acceptance of $r\rho$. Therefore, we introduce a new rule of the form $r'' = X \stackrel{K}{\to} Z_F$ such that $K = \Upsilon^f_M(r\rho)$ and $\Upsilon^f_{M_{\mathrm{PAR}}}(r'') = K$. M_{PAR} is formally defined as follows.

Definition 6. The *MBRS* $M_{\mathrm{PAR}} = \langle \Re_{\mathrm{PAR}}, \langle \Re_{\mathrm{PAR},1}, \ldots, \Re_{\mathrm{PAR},n} \rangle \rangle$ is the least parallel *MBRS* over $Var \cup \{Z_F\}$ and the alphabet $\Sigma \cup P_n$ such that:

1. $\Re_{\mathrm{PAR}} \supseteq \Re_P$ and $\Re_{\mathrm{PAR},i} \supseteq \Re_{P,i}$ for all $i = 1, \ldots, n$.
2. Let $r = X \stackrel{a}{\to} Z.Y \in \Re$, $Z \stackrel{\sigma}{\Rightarrow}_{\Re_{\mathrm{PAR}}} p$ for some term p, and $K = \Upsilon^f_M(r) \cup \Upsilon^f_{M_{\mathrm{PAR}}}(\sigma)$. Then, $r' = X \stackrel{K}{\to} Z_F \in \Re_{\mathrm{PAR}}$ and $\Upsilon^f_{M_{\mathrm{PAR}}}(r') = K$.
3. Let $r = X \stackrel{a}{\to} Z.Y \in \Re$, $Z \stackrel{\sigma}{\Rightarrow}_{\Re_{\mathrm{PAR}}} \varepsilon$, and $K = \Upsilon^f_M(r) \cup \Upsilon^f_{M_{\mathrm{PAR}}}(\sigma)$. Then, $r' = X \stackrel{K}{\to} Y \in \Re_{\mathrm{PAR}}$ and $\Upsilon^f_{M_{\mathrm{PAR}}}(r') = K$.
4. Let $r = X \stackrel{a}{\to} Z.Y \in \Re$, $r' = W.Y \stackrel{b}{\to} W' \in \Re$, $Z \stackrel{\sigma}{\Rightarrow}_{\Re_{\mathrm{PAR}}} W$, and $K = \Upsilon^f_M(rr') \cup \Upsilon^f_{M_{\mathrm{PAR}}}(\sigma)$. Then, $r'' = X \stackrel{K}{\to} W' \in \Re_{\mathrm{PAR}}$ and $\Upsilon^f_{M_{\mathrm{PAR}}}(r'') = K$.

Note that Property 2 in Definition 6 corresponds to case **A** seen above, while Property 3 (resp., Property 4) corresponds to case **B** (resp., case **C**).

Lemma 1. *The parallel* MBRS M_{PAR} *can be effectively constructed.*

Proof. Fig. 1 reports the procedure BUILD-PARALLEL-MBRS(M) that builds M_{PAR}. The algorithm uses the routine *UPDATE*(r', K) defined as

$\Re_{\mathrm{PAR}} := \Re_{\mathrm{PAR}} \cup \{r'\}$;
for each $i \in K$ **do** $\Re_{\mathrm{PAR},i} := \Re_{\mathrm{PAR},i} \cup \{r'\}$;

Note that by Proposition 4, the conditions in each of the **if** statements in lines 7, 9 and 13 are decidable, therefore, the procedure is effective. Moreover, since the set of rules of the form $X \stackrel{K}{\to} Y$ with $X \in Var$, $Y \in Var \cup \{Z_F\}$ and $K \in P_n$ is finite, termination is guaranteed. □

The following two Lemmata 2 and 3 establish the correctness of our construction. The proof of Lemma 2 has been already sketched above. Therefore, we prove only Lemma 3.

Lemma 2. *Let* $p \stackrel{\sigma}{\Rightarrow}_{\Re} t \| p'$ *with* $p, p' \in T_{\mathrm{PAR}}$. *Then, there exists* $s \in T_{\mathrm{PAR}}$ *such that* $p \stackrel{\rho}{\Rightarrow}_{\Re_{\mathrm{PAR}}} s \| p'$, $\Upsilon^f_{M_{\mathrm{PAR}}}(\rho) = \Upsilon^f_M(\sigma)$, $s = \varepsilon$ *if* $t = \varepsilon$, *and* $|\rho| > 0$ if $|\sigma| > 0$.

Lemma 3. *Let* $p \stackrel{\sigma}{\Rightarrow}_{\Re_{\mathrm{PAR}}} p' \| p''$ *such that* $p, p', p'' \in T_{\mathrm{PAR}}$, p' *does not contain occurrences of* Z_F, *and* p'' *does not contain occurrences of variables in Var. Then, there exists* $t \in T$ *such that* $p \stackrel{\rho}{\Rightarrow}_{\Re} p' \| t$, $\Upsilon^f_M(\rho) = \Upsilon^f_{M_{\mathrm{PAR}}}(\sigma)$, $t = \varepsilon$ *if* $p'' = \varepsilon$, *and* $|\rho| > 0$ *if* $|\sigma| > 0$.

Proof. If $\Re_{\mathrm{PAR}} = \Re_P$, then the assertion is obvious. Thus, assume that $\Re_{\mathrm{PAR}} \backslash \Re_P = \{r_1, \ldots, r_m\}$ with $m \geqslant 1$, where for any $i = 1, \ldots, m$, r_i is the ith rule added to $\Re_{\mathrm{PAR}}$ during the **repeat** loop of the algorithm of Lemma 1. For any $i = 1, \ldots, m$, let us denote by M^i_{PAR} (with support $\Re^i_{\mathrm{PAR}}$) the parallel *MBRS* M_{PAR} *soon after* the rule r_i is added during

Algorithm BUILD–PARALLEL–MBRS(M)

1 $\Re_{PAR} := \Re_P$;
2 **for** $i = 1, \ldots, n$ **do** $\Re_{PAR,i} := \Re_{P,i}$;
3 **repeat**
4 *flag:=false;*
5 **for each** $r = X \xrightarrow{a} Z.Y \in \Re$ *and* $K_1 \in P_n$ **do**
6 *Set* $K = K_1 \cup \Upsilon_M^f(r)$;
7 **if** $Z \overset{\sigma}{\Rightarrow}_{\Re_{PAR}} p$ *for some p such that* $\Upsilon^f_{M_{PAR}}(\sigma) = K_1$ **then**
8 **if** $r' = X \xrightarrow{K} Z_F \notin \Re_{PAR}$ **then** $UPDATE(r', K)$; *flag:=true;*
9 **if** $Z \overset{\sigma}{\Rightarrow}_{\Re_{PAR}} \varepsilon$ *such that* $\Upsilon^f_{M_{PAR}}(\sigma) = K_1$ **then**
10 **if** $r' = X \xrightarrow{K} Y \notin \Re_{PAR}$ **then** $UPDATE(r', K)$; *flag:=true;*
11 **for each** $r' = W.Y \xrightarrow{b} W' \in \Re$ **do**
12 *Set* $K = K_1 \cup \Upsilon^f_M(rr')$;
13 **if** $Z \overset{\sigma}{\Rightarrow}_{\Re_{PAR}} W$ *such that* $\Upsilon^f_{M_{PAR}}(\sigma) = K_1$ **then**
14 **if** $r'' = X \xrightarrow{K} W' \notin \Re_{PAR}$ **then** $UPDATE(r'', K)$; *flag:=true;*
15 **until** *flag = false*

Fig. 1. Algorithm to build the parallel *MBRS* M_{PAR}.

the computation. Moreover, let $M^0_{\text{PAR}} = M_P$. Then, since $M_{\text{PAR}} = M^m_{\text{PAR}}$, it is sufficient to prove the assertion with M_{PAR} replaced with M^i_{PAR} for each $i = 0, \ldots, m$. The assertion is obvious for $i = 0$. Now, we consider the induction step ($i > 0$). Let d be a derivation in M^i_{PAR} of the form $p \overset{\sigma}{\Rightarrow} p' \| p''$ such that $p, p', p'' \in T_{\text{PAR}}$, p' does not contain occurrences of Z_F, and p'' does not contain occurrences of variables in *Var*. We proceed by induction on $|\sigma|$. For $|\sigma| = 0$, the assertion is obvious. Now, let us assume that $|\sigma| > 0$. In this case the derivation d can be written in the form

$$p \overset{\sigma'}{\Rightarrow} \overline{p}' \| \overline{p}'' \overset{r}{\Rightarrow} p' \| p'' \quad \text{with } |\sigma'| < |\sigma|,\ r \in \Re^i_{\text{PAR}} \text{ and } \overline{p}', \overline{p}'' \in T_{\text{PAR}}$$

Moreover, $\overline{p}'$ does not contain occurrences of Z_F, and $\overline{p}''$ does not contain occurrences of variables in *Var*. By the induction hypothesis, there exists $\overline{t} \in T$ such that $p \overset{\rho'}{\Rightarrow}_{\Re} \overline{p}' \| \overline{t}$, $\Upsilon^f_M(\rho') = \Upsilon^f_{M^i_{\text{PAR}}}(\sigma')$, and $\overline{t} = \varepsilon$ if $\overline{p}'' = \varepsilon$. There are two cases:

1. r is a PAR rule of $\Re$. By construction $\Upsilon^f_M(r) = \Upsilon^f_{M^i_{\text{PAR}}}(r)$. Moreover, $\overline{p}'' = p''$ and $\overline{p}' \overset{r}{\Rightarrow}_{\Re} p'$. Then, we deduce that $p \overset{\rho'}{\Rightarrow}_{\Re} \overline{p}' \| \overline{t} \overset{r}{\Rightarrow}_{\Re} p' \| \overline{t}$, $\Upsilon^f_M(\rho' r) = \Upsilon^f_{M^i_{\text{PAR}}}(\sigma' r)$, and $\overline{t} = \varepsilon$ if $p'' = \varepsilon$. Thus, the assertion holds.
2. $r = X \xrightarrow{K} X' \in \Re^i_{\text{PAR}} \setminus \Re_P$ $(= \{r_1, \ldots, r_i\})$. Assume that $X' \in Var$ (the other case with $X' = Z_F$ being similar). By construction $\Upsilon^f_{M^i_{\text{PAR}}}(r) = K$. Moreover, $\overline{p}'' = p''$ and there is $p_1 \in T_{\text{PAR}}$ such that $\overline{p}' = p_1 \| X$ and $p' = p_1 \| X'$.
   Now, we claim that there exists a derivation of the form $X \overset{\rho''}{\Rightarrow}_{\Re} X'$ such that $\Upsilon^f_M(\rho'') = K$ and $|\rho''| > 0$. First, we note that the assertion easily follows from this claim. Indeed, we deduce that $p \overset{\rho'}{\Rightarrow}_{\Re} \overline{p}' \| \overline{t} \overset{\rho''}{\Rightarrow}_{\Re} p' \| \overline{t}$, $\Upsilon^f_M(\rho' \rho'') = \Upsilon^f_{M^i_{\text{PAR}}}(\sigma' r)$, and $\overline{t} = \varepsilon$ if $p'' = \varepsilon$. It remains to prove the claim. Let us consider the algorithm of Lemma 1. The rule r is added to $\Re_{\text{PAR}}$ during an iteration of the **for** loop in lines 5–14, in which a rule $r' \in \Re$ of the form $X \xrightarrow{a} Z.Y$ is examined. Since $X' \in Var$, r is added to $\Re_{\text{PAR}}$ either in line 10 or in line 14. Let us consider the latter case (the former case being similar). Then, r is added to $\Re_{\text{PAR}}$ by the inner **for** loop in lines 11–14 when a rule $r'' \in \Re$ of the form $W.Y \xrightarrow{b} X'$ is examined. Moreover, the condition of the **if** statement in line 13 is satisfied: there is a finite derivation in M_{PAR} of the form $Z \overset{\pi}{\Rightarrow}_{\Re_{\text{PAR}}} W$ such that $\Upsilon^f_{M_{\text{PAR}}}(\pi) \cup \Upsilon^f_M(r' r'') = K$. Since in this computation phase $M_{\text{PAR}} = M^j_{\text{PAR}}$ with $j < i$, by the induction hypothesis there exists a finite derivation in $\Re$ of the form $Z \overset{\rho}{\Rightarrow}_{\Re} W$ such that $\Upsilon^f_M(\rho) = \Upsilon^f_{M_{\text{PAR}}}(\pi)$. Therefore, there is a derivation in $\Re$ of the form $X \overset{r'}{\Rightarrow}_{\Re} Z.Y \overset{\rho}{\Rightarrow}_{\Re} W.Y \overset{r''}{\Rightarrow}_{\Re} X'$ with $\Upsilon^f_M(r' \rho r'') = \Upsilon^f_M(r' r'') \cup \Upsilon^f_{M_{\text{PAR}}}(\pi) = K$. This completes the proof of the claim. $\square$

Now, we can prove the main result of this subsection.

Theorem 2. *The Finite Acceptance Problem is decidable.*

Proof. Let $X, Y \in Var$ and $K \in P_n$. By Lemmata 2 and 3 there exists a finite derivation in M of the form $X \stackrel{\sigma}{\Rightarrow}$ (resp., $X \stackrel{\sigma}{\Rightarrow} Y$) such that $\Upsilon_M^f(\sigma) = K$ if, and only if, there exists a finite derivation in the parallel *MBRS* M_{PAR} of the form $X \stackrel{\rho}{\Rightarrow}_{\Re_{\mathrm{PAR}}}$ (resp., $X \stackrel{\rho}{\Rightarrow}_{\Re_{\mathrm{PAR}}} Y$) such that $\Upsilon_{M_{\mathrm{PAR}}}^f(\rho) = K$. Then, the result directly follows from Proposition 4. □

4.2. Step 2 (Fairness problem)

In this subsection we solve the Fairness Problem for the *MBRS* in normal form M and a given pair of sets $K, K_\omega \in P_n$. In the decidability proof, we use the parallel *MBRS* M_{PAR} computed by the algorithm of Lemma 1.

For technical reasons, we define a class of derivations, in symbols $\Pi(K, K_\omega)$, that is the set of derivations d in $\Re$ such that there is *not* a subderivation of d that is a (K, K_ω)-accepting infinite derivation in M. Our first goal is to show that we can limit ourselves to consider only this class of derivations. More precisely, we show how to build effectively, starting from M and M_{PAR}, a sequential *MBRS* such that, except for decidable questions on this sequential *MBRS*, the Fairness Problem for M and the pair $K, K_\omega \in P_n$ is reduced to check the existence of a (K, K_ω)-accepting infinite derivation in M starting from a variable and belonging to the class $\Pi(K, K_\omega)$.
In order to illustrate this, let d be a (K, K_ω)-accepting infinite derivation in M from a variable X. If d does *not* belong to $\Pi(K, K_\omega)$, then it can be written in the form $X \stackrel{\rho}{\Rightarrow} t\|W \stackrel{r}{\Rightarrow} t\|(Z.Y) \stackrel{\nu}{\Rightarrow}$, with $Z \in Var$ and $r = W \stackrel{a}{\rightarrow} Z.Y$, and such that there exists a subderivation of $t\|(Z.Y) \stackrel{\nu}{\Rightarrow}$ from Z that is a (K, K_ω)-accepting infinite derivation in M. Following this argument, we can prove that there exist $m \in \mathbb{N}\setminus\{0\} \cup \{\infty\}$, a sequence of variables $(X_h)_{h=0}^{h=m}$ with $X_0 = X$, and a sequence of SEQ rules $(r_h)_{h=1}^{h=m}$ such that one of the following two conditions is satisfied:

1. m is finite, for each $h = 0, \ldots, m-1$, we have that $X_h \stackrel{\rho_h}{\Rightarrow} t_h\|Y_{h+1}$, $r_{h+1} = Y_{h+1} \stackrel{a_{h+1}}{\rightarrow} X_{h+1}.Z_{h+1}$, $\Upsilon_M^f(\rho_h r_{h+1}) \subseteq K$, and there exists a (K, K_ω)-accepting infinite derivation in M from X_m *belonging* to $\Pi(K, K_\omega)$.
2. (for $K = K_\omega$) m is infinite, and for all $h \in \mathbb{N}$ we have that $X_h \stackrel{\rho_h}{\Rightarrow} t_h\|Y_{h+1}$, $r_{h+1} = Y_{h+1} \stackrel{a_{h+1}}{\rightarrow} X_{h+1}.Z_{h+1}$, and $\Upsilon_M^f(\rho_0 r_1 \rho_1 r_2 \ldots) = \Upsilon_M^\infty(\rho_0 r_1 \rho_1 r_2 \ldots) = K_\omega$.

For each h, let us consider the derivation $X_h \stackrel{\rho_h}{\Rightarrow} t_h\|Y_{h+1}$. By Lemma 2 there exists a finite derivation in M_{PAR} of the form $X_h \stackrel{\lambda_h}{\Rightarrow}_{\Re_{\mathrm{PAR}}} p_h\|Y_{h+1}$ such that $\Upsilon_{M_{\mathrm{PAR}}}^f(\lambda_h) = \Upsilon_M^f(\rho_h)$ and $p_h \in T_{\mathrm{PAR}}$. By Proposition 4 for each $K' \in P_n$, it is decidable whether variable Y_{h+1} is partially reachable in M_{PAR} from X_h through a derivation having finite acceptance K'. The idea is to introduce a rule of the form $X_h \stackrel{K'}{\rightarrow} Y_{h+1}$ where $K' = \Upsilon_{M_{\mathrm{PAR}}}^f(\lambda_h)$, and whose finite acceptance is K'. Let us denote by M_{SEQ} the *sequential MBRS* (with n accepting components) containing these new rules (whose number is finite) and all the SEQ rules of M having the form $X \stackrel{a}{\rightarrow} Z.Y$, and whose accepting components agree with the labels of the new rules. Then, case 2 above amounts to check the existence of a (K, K_ω)-accepting infinite derivation in M_{SEQ} from variable X. By Proposition 6 this is decidable. Case 1 amounts to check the existence of a variable $Y \in Var$ such that Y is s-reachable from X in M_{SEQ} through a derivation with finite acceptance (in M_{SEQ}) $K' \subseteq K$ (by Proposition 6 this is decidable), and there exists a (K, K_ω)-accepting infinite derivation in M from Y belonging to $\Pi(K, K_\omega)$. M_{SEQ} is formally defined as follows.

Definition 7. By $M_{\mathrm{SEQ}} = \langle \Re_{\mathrm{SEQ}}, \langle \Re_{\mathrm{SEQ},1}, \ldots, \Re_{\mathrm{SEQ},n}\rangle\rangle$ we denote the *sequential MBRS* over *Var* and the alphabet $\Sigma \cup P_n$ defined as follows:

- $\Re_{\mathrm{SEQ}} = \{X \stackrel{a}{\rightarrow} Z.Y \in \Re\} \cup \{X \stackrel{K'}{\rightarrow} Y \mid X, Y \in Var,\ X \stackrel{\sigma}{\Rightarrow}_{\Re_{\mathrm{PAR}}} p\|Y$ for some $p \in T_{\mathrm{PAR}}, |\sigma| > 0,$ and $\Upsilon_{M_{\mathrm{PAR}}}^f(\sigma) = K'\}$.
- $\Re_{\mathrm{SEQ},i} = \{X \stackrel{a}{\rightarrow} Z.Y \in \Re_i\} \cup \{X \stackrel{K'}{\rightarrow} Y \in \Re_{\mathrm{SEQ}} \mid i \in K'\}$ for all $i = 1, \ldots, n$.

By Proposition 4 we obtain the following result.

Lemma 4. *M_{SEQ} can be built effectively.*

Thus, we obtain a first reduction of the Fairness Problem.

Lemma 5. *Given $X \in Var$, there exists a (K, K_ω)-accepting infinite derivation in M from X if, and only if, one of the following conditions is satisfied*:
1. *There is a variable $Y \in Var$ such that Y is s-reachable from X in $\Re_{\text{SEQ}}$ through a $(K', \emptyset)$-accepting derivation in M_{SEQ} with $K' \subseteq K$, and there is a (K, K_ω)-accepting infinite derivation in M from Y belonging to $\Pi(K, K_\omega)$.*
2. (*Only when $K = K_\omega$*) *There exists a (K, K_ω)-accepting infinite derivation in M_{SEQ} from X.*

Proof. The complete proof is given in Appendix A.1.

Therefore, it remains to manage the class $\Pi(K, K_\omega)$. We proceed by induction on $|K| + |K_\omega|$. Let $d = (X \overset{\sigma}{\Rightarrow})$ be a (K, K_ω)-accepting infinite derivation in M from $X \in Var$ belonging to $\Pi(K, K_\omega)$. If $|K| + |K_\omega| = 0$ (i.e., $K = K_\omega = \emptyset$), then we deduce that all the subderivations of d must be finite (since the rule sequence associated with such a subderivation is a subsequence of σ). By Section 4.1 we can keep track of the meaningful information of such subderivations (together with the "caller" rules having the form $W \overset{a}{\to} Y.Z$ and possibly with the "return" rules, if any, having the form $Y'.Z \overset{b}{\to} W'$) by using the rules of the parallel *MBRS* M_{PAR}. This means that d can be simulated (preserving the finite and infinite acceptance) in M_{PAR} (and vice versa). Therefore, for $K = K_\omega = \emptyset$, by Lemma 5 and Propositions 5–6 it follows that the Fairness Problem is decidable. Now, assume that $|K| + |K_\omega| > 0$. By the induction hypothesis for any $\overline{K}, \overline{K}_\omega \in P_n$ such that $|\overline{K}| + |\overline{K}_\omega| < |K| + |K_\omega|$, the Fairness Problem with input $\overline{K}$ and $\overline{K}_\omega$ is decidable. Since d belongs to $\Pi(K, K_\omega)$, the idea is to keep track of the infinite subderivations $d' = (Y \overset{\rho}{\Rightarrow}_\Re)$ of d starting at process variables[6] and activated by "caller" rules r of the form $W \overset{a}{\to} Y.Z$ by using new PAR rules r' of the form $W \overset{K', K'_\omega}{\to} Z_F$ where K' (resp., K'_ω) is the finite (resp., the infinite) acceptance of $r\rho$ in M. Formally, we define two extensions of M_{PAR} (with the same support) that will contain these new PAR rules r'. The accepting components of the first (resp., the second) extension agree with the first component K' (resp., the second component K'_ω) of the label of r'.

Definition 8. By $M_{\text{PAR}}^{K,K_\omega} = \langle \Re_{\text{PAR}}^{K,K_\omega}, \langle \Re_{\text{PAR},1}^{K,K_\omega}, \ldots, \Re_{\text{PAR},n}^{K,K_\omega} \rangle\rangle$ and $M_{\text{PAR},\infty}^{K,K_\omega} = \langle \Re_{\text{PAR}}^{K,K_\omega}, \langle \Re_{\text{PAR},\infty,1}^{K,K_\omega}, \ldots, \Re_{\text{PAR},\infty,n}^{K,K_\omega} \rangle\rangle$ we denote the parallel *MBRS* over $Var \cup \{Z_F\}$ and the alphabet $\Sigma \cup P_n \cup P_n \times P_n$ (with the same support), defined as follows:

- $\Re_{\text{PAR}}^{K,K_\omega} = \Re_{\text{PAR}} \cup \{X \overset{\overline{K}, \overline{K}_\omega}{\to} Z_F \mid \overline{K} \subseteq K, \overline{K}_\omega \subseteq K_\omega,$ there exists $r = X \overset{a}{\to} Z.Y \in \Re$ and an infinite derivation $Z \overset{\sigma}{\Rightarrow}_\Re$ such that $|\Upsilon_M^f(\sigma)| + |\Upsilon_M^\infty(\sigma)| < |K| + |K_\omega|$ and $\Upsilon_M^f(\sigma) \cup \Upsilon_M^f(r) = \overline{K}$ and $\Upsilon_M^\infty(\sigma) = \overline{K}_\omega\}$.
- $\Re_{\text{PAR},i}^{K,K_\omega} = \Re_{\text{PAR},i} \cup \{X \overset{\overline{K}, \overline{K}_\omega}{\to} Z_F \in \Re_{\text{PAR}}^{K,K_\omega} \mid i \in \overline{K}\}$ for all $i = 1, \ldots, n$.
- $\Re_{\text{PAR},i,\infty}^{K,K_\omega} = \{X \overset{\overline{K}, \overline{K}_\omega}{\to} Z_F \in \Re_{\text{PAR}}^{K,K_\omega} \mid i \in \overline{K}_\omega\}$ for all $i = 1, \ldots, n$.

Note that for $K = K_\omega = \emptyset$, $M_{\text{PAR}}^{K,K_\omega}$ coincides with M_{PAR} and each accepting component of $M_{\text{PAR},\infty}^{K,K_\omega}$ coincides with the empty set.

The following two Lemmata 6 and 7 establish the correctness of our construction.

Lemma 6. *Let $p \overset{\sigma}{\Rightarrow}$ be a $(\overline{K}, \overline{K}_\omega)$-accepting non-null derivation in M from $p \in T_{\text{PAR}}$ belonging to $\Pi(K, K_\omega)$, where $\overline{K} \subseteq K$ and $\overline{K}_\omega \subseteq K_\omega$. Then, there exists in $\Re_{\text{PAR}}^{K,K_\omega}$ a derivation of the form $p \overset{\rho}{\Rightarrow}$ such that $\Upsilon^f_{M_{\text{PAR}}^{K,K_\omega}}(\rho) = \overline{K}$, $\Upsilon^\infty_{M_{\text{PAR}}^{K,K_\omega}}(\rho) \cup \Upsilon^f_{M_{\text{PAR},\infty}^{K,K_\omega}}(\rho) = \overline{K}_\omega$. Moreover, if σ is infinite, then either ρ is infinite or contains some occurrence of rule in $\Re_{\text{PAR}}^{K,K_\omega} \setminus \Re_{\text{PAR}}$.*

[6] Note that $\Upsilon_M^f(\rho) \subseteq K$, $\Upsilon_M^\infty(\rho) \subseteq K_\omega$ and $|\Upsilon_M^f(\rho)| + |\Upsilon_M^\infty(\rho)| < |K| + |K_\omega|$.

Proof. First, we prove the following property.

A There exist $p' \in T_{\mathrm{PAR}}$, a non-empty finite rule sequence λ in $\Re_{\mathrm{PAR}}^{K,K_\omega}$, and a non-empty subsequence η (possibly infinite) of σ such that $\min(pr(\eta)) = \min(pr(\sigma))$ (i.e., the first rule occurrence in η is the first rule occurrence in σ), $p \overset{\lambda}{\Rightarrow}_{\Re_{\mathrm{PAR}}^{K,K_\omega}} p'$, $\Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR}}}(\lambda) = \Upsilon^f_M(\eta)$, $\Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR},\infty}}(\lambda) = \Upsilon^\infty_M(\eta)$, and there exists a derivation d' of the form $p' \overset{\sigma\backslash\eta}{\Rightarrow}_\Re$ belonging to $\Pi(K, K_\omega)$. Moreover, if σ is infinite, then either $\sigma\backslash\eta$ is infinite or λ is a rule in $\Re_{\mathrm{PAR}}^{K,K_\omega}\backslash\Re_{\mathrm{PAR}}$.

The derivation $p \overset{\sigma}{\Rightarrow}_\Re$ can be rewritten as $p \overset{r}{\Rightarrow}_\Re t \overset{\sigma'}{\Rightarrow}_\Re$. First, assume that r is a PAR rule. Then, $t \in T_{\mathrm{PAR}}$ and, by construction, $r \in \Re_{\mathrm{PAR}}$, $\Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR}}}(r) = \Upsilon^f_{M_{\mathrm{PAR}}}(r) = \Upsilon^f_M(r)$, and $\Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR},\infty}}(r) = \emptyset = \Upsilon^\infty_M(r)$. Since $t \overset{\sigma'}{\Rightarrow}_\Re$ belongs to $\Pi(K, K_\omega)$, property **A** follows, setting $p' = t$, $\lambda = r$ and $\eta = r$. If r is not a PAR rule, then r is a SEQ rule of the form $Z \overset{a}{\to} Z'.Y$. Thus, $p = p''\|Z$ and $t = p''\|(Z'.Y)$ with $p'' \in T_{\mathrm{PAR}}$. Let $Z' \overset{v}{\Rightarrow}_\Re$ be a subderivation of $t = p''\|(Z'.Y) \overset{\sigma'}{\Rightarrow}_\Re$ from Z'. By definition of subderivation we can distinguish three subcases:

(i) $p'' \overset{\sigma'\backslash v}{\Rightarrow}_\Re$ and this derivation belongs to $\Pi(K, K_\omega)$.

(ii) $Z' \overset{v}{\Rightarrow}_\Re$ leads to ε and $p''\|(Z'.Y) \overset{\sigma'}{\Rightarrow}_\Re$ can be written as $p''\|(Z'.Y) \overset{\sigma_1}{\Rightarrow} t'\|Y \overset{\sigma_2}{\Rightarrow}$, with $p'' \overset{\sigma'_1}{\Rightarrow} t'$ and $\sigma_1 \in$ *Interleave*(v, σ'_1). Moreover, $p''\|Y \overset{\sigma'_1}{\Rightarrow} t'\|Y \overset{\sigma_2}{\Rightarrow}$ and this derivation belongs to $\Pi(K, K_\omega)$.

(iii) $Z' \overset{v}{\Rightarrow}_\Re$ leads to a variable $W \in Var$ and $p''\|(Z'.Y) \overset{\sigma'}{\Rightarrow}_\Re$ can be written as $p''\|(Z'.Y) \overset{\sigma_1}{\Rightarrow} t'\|(W.Y) \overset{r'}{\Rightarrow} t'\|W' \overset{\sigma_2}{\Rightarrow}$

$$\text{with } p'' \overset{\sigma'_1}{\Rightarrow} t',\ r' = W.Y \overset{b}{\to} W' \text{ and } \sigma_1 \in \mathit{Interleave}(v, \sigma'_1). \tag{6}$$

Moreover, $p''\|W' \overset{\sigma'_1}{\Rightarrow} t'\|W' \overset{\sigma_2}{\Rightarrow}$ and this derivation belongs to $\Pi(K, K_\omega)$.

Since cases (ii)–(iii) are similar, we consider only cases (i) and (iii). First, let us consider case (i). Assume that $Z' \overset{v}{\Rightarrow}_\Re$ is infinite (the other case being similar). Since, by hypothesis, $\Upsilon^f_M(v) \subseteq K$, $\Upsilon^\infty_M(v) \subseteq K_\omega$, $|\Upsilon^f_M(v)| + |\Upsilon^\infty_M(v)| < |K| + |K_\omega|$ and $\Upsilon^f_M(r) \subseteq K$, by the definition of $\Re_{\mathrm{PAR}}^{K,K_\omega}$, it follows that $r'' = Z \overset{K',K'_\omega}{\to} Z_F \in \Re_{\mathrm{PAR}}^{K,K_\omega}\backslash\Re_{\mathrm{PAR}}$ where $K' = \Upsilon^f_M(rv)$, $K'_\omega = \Upsilon^\infty_M(rv)$, $\Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR}}}(r'') = K'$, and $\Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR},\infty}}(r'') = K'_\omega$. Hence, it holds that $p = p''\|Z \overset{r''}{\Rightarrow}_{\Re_{\mathrm{PAR}}^{K,K_\omega}} p''\|Z_F$. Moreover, $p''\|Z_F \overset{\sigma'\backslash v}{\Rightarrow}_\Re$ and this derivation belongs to $\Pi(K, K_\omega)$. Since $\sigma'\backslash v = \sigma\backslash rv$ property **A** follows, setting $p' = p''\|Z_F$, $\lambda = r''$ and $\eta = rv$. Now, consider case (iii). Since $Z \overset{r}{\Rightarrow}_\Re Z'.Y \overset{v}{\Rightarrow}_\Re W.Y \overset{r'}{\Rightarrow}_\Re W'$, by Lemma 2 it follows that $Z \overset{\lambda}{\Rightarrow}_{\Re_{\mathrm{PAR}}} W'$ with $\Upsilon^f_{M_{\mathrm{PAR}}}(\lambda) = \Upsilon^f_M(rvr')$. By construction, $\Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR}}}(\lambda) = \Upsilon^f_{M_{\mathrm{PAR}}}(\lambda)$ and $\Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR},\infty}}(\lambda) = \emptyset = \Upsilon^\infty_M(rvr')$. Since $\sigma\backslash rvr' = \sigma'_1\sigma_2$, property **A** follows setting $p' = p''\|W'$ and $\eta = rvr'$.

Therefore, Property **A** holds. Since $\sigma\backslash\eta$ is a subsequence of σ, we have $\Upsilon^f_M(\sigma\backslash\eta) \subseteq K$ and $\Upsilon^\infty_M(\sigma\backslash\eta) \subseteq K_\omega$. Thus, if $\sigma \neq \eta$ we can apply property **A** to the derivation d' (of the form $p' \overset{\sigma\backslash\eta}{\Rightarrow}_\Re$). Repeating this argument it follows that there exists $m \in \mathbb{N} \cup \{\infty\}$, a sequence $(p_h)_{h=0}^{h=m+1}$ of terms in T_{PAR}, a sequence $(\lambda_h)_{h=0}^{h=m}$ of non-empty finite rule sequences in $\Re_{\mathrm{PAR}}^{K,K_\omega}$, two sequences $(\sigma_h)_{h=0}^{h=m}$ and $(\eta_h)_{h=0}^{h=m}$ of non-empty rule sequences in $\Re$ such that for all $h = 0, \ldots, m$

1. $p = p_0$ and $\sigma = \sigma_0$.
2. η_h is a subsequence of σ_h, $\min(pr(\eta_h)) = \min(pr(\sigma_h))$, and if $h \neq m$ then $\sigma_{h+1} = \sigma_h\backslash\eta_h$.
3. $p_h \overset{\lambda_h}{\Rightarrow}_{\Re_{\mathrm{PAR}}^{K,K_\omega}} p_{h+1}$, $\Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR}}}(\lambda_h) = \Upsilon^f_M(\eta_h)$, $\Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR},\infty}}(\lambda_h) = \Upsilon^\infty_M(\eta_h)$, and $p_h \overset{\sigma_h}{\Rightarrow}_\Re$.
4. If m is finite, then $\sigma_m = \eta_m$. If σ is infinite, then either m is infinite or λ_h is a rule in $\Re_{\mathrm{PAR}}^{K,K_\omega}\backslash\Re_{\mathrm{PAR}}$ for some h.

By setting $\rho = \lambda_0\lambda_1\ldots$ we have that $p \overset{\rho}{\Rightarrow}_{\Re_{\mathrm{PAR}}^{K,K_\omega}}$. By Property 4 it follows that if σ is infinite, then either ρ is infinite or ρ contains some occurrence of rule in $\Re_{\mathrm{PAR}}^{K,K_\omega}\backslash\Re_{\mathrm{PAR}}$. Let us assume that $m = \infty$. The proof for m finite is simpler. By Properties 1–2 $\eta_0, \eta_1, \ldots$ are non-empty subsequences of σ two by two disjoints. Since σ is infinite, we can assume that $pr(\sigma) = \mathbb{N}$. Now, let us show that

5. $\sigma \in$ *Interleave*$((\eta_h)_{h\in\mathbb{N}})$.

By Proposition 1 it is sufficient to prove that for all $h \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $h \in pr(\eta_i)$. By Property 2 it follows that, for all $h \in \mathbb{N}$, $\min(pr(\sigma_h)) < \min(pr(\sigma_{h+1}))$. Let $h \in \mathbb{N}$, then there exists the smallest $i \in \mathbb{N}$ such that $h \notin pr(\sigma_i)$. Since $\sigma_0 = \sigma$, we have that $i > 0$ and $h \in pr(\sigma_{i-1})$. Since $\sigma_i = \sigma_{i-1} \backslash \eta_{i-1}$, $h \notin pr(\sigma_i)$ and $h \in pr(\sigma_{i-1})$, it follows that $h \in pr(\eta_{i-1})$. Thus, Property 5 holds. By Properties 3, 5, and Proposition 3 we have that $\Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR}}}(\rho) = \bigcup_{h\in\mathbb{N}} \Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR}}}(\lambda_h) = \bigcup_{h\in\mathbb{N}} \Upsilon^f_M(\eta_h) = \Upsilon^f_M(\sigma) = \overline{K}$. Moreover,

$$\begin{aligned}\Upsilon^\infty_{M^{K,K_\omega}_{\mathrm{PAR}}}(\rho) \cup \Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR},\infty}}(\rho) &= \bigoplus_{h\in\mathbb{N}} \Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR}}}(\lambda_h) \cup \bigcup_{h\in\mathbb{N}} \Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR},\infty}}(\lambda_h)\\ &= \bigoplus_{h\in\mathbb{N}} \Upsilon^f_M(\eta_h) \cup \bigcup_{h\in\mathbb{N}} \Upsilon^\infty_M(\eta_h) = \Upsilon^\infty_M(\sigma) = \overline{K}_\omega.\end{aligned}$$

This concludes the proof. □

Lemma 7. *Let* $p \overset{\sigma}{\Rightarrow}_{\Re^{K,K_\omega}_{\mathrm{PAR}}}$ *such that* $p \in T_{\mathrm{PAR}}$, *and* σ *is either infinite or contains some occurrence of rule in* $\Re^{K,K_\omega}_{\mathrm{PAR}} \backslash \Re_{\mathrm{PAR}}$. *Then, there exists in* $\Re$ *an infinite derivation of the form* $p \overset{\delta}{\Rightarrow}$ *such that* $\Upsilon^f_M(\delta) = \Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR}}}(\sigma)$ *and* $\Upsilon^\infty_M(\delta) = \Upsilon^\infty_{M^{K,K_\omega}_{\mathrm{PAR}}}(\sigma) \cup \Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR},\infty}}(\sigma)$.

Proof. The complete proof is given in Appendix A.2.

Finally, we can prove the desired result.

Theorem 3. *The Fairness Problem is decidable for* MBRS *in normal form.*

Proof. We start by constructing M_{PAR} and M_{SEQ} (they do not depend on K and K_ω). Then, we accumulate information about the existence of $(\overline{K}, \overline{K}_\omega)$-accepting infinite derivations in M from variables in *Var*, where $|\overline{K}|+|\overline{K}_\omega| \leqslant |K|+|K_\omega|$ and $\overline{K} \subseteq K$ and $\overline{K}_\omega \subseteq K_\omega$, proceeding for crescent values of $|\overline{K}|+|\overline{K}_\omega|$. We keep track of this information by adding new PAR rules according to Definition 8. For $|\overline{K}|+|\overline{K}_\omega| = 0$, since $M^{\emptyset,\emptyset}_{\mathrm{PAR}}$ coincides with M_{PAR}, by Lemmata 5, 6, and 7 we obtain the following decidable (by Propositions 4 and 6) characterization for the existence of a $(\emptyset, \emptyset)$-accepting infinite derivation in M from a variable X:

- Either (1) there exists a $(\emptyset, \emptyset)$-accepting infinite derivation in M_{SEQ} from X, or (2) there exists a variable Y s-reachable from X in M_{SEQ} through a derivation having finite acceptance (in M_{SEQ}) $K' = \emptyset$, and there exists a $(\emptyset, \emptyset)$-accepting infinite derivation in M_{PAR} from Y.

When $|\overline{K}| + |\overline{K}_\omega| > 0$ (assuming without loss of generality that $\overline{K} = K$ and $\overline{K}_\omega = K_\omega$), then by the induction hypothesis the parallel *MBRS* $M^{K,K_\omega}_{\mathrm{PAR}}$ and $M^{K,K_\omega}_{\mathrm{PAR},\infty}$ can be built effectively. Therefore, by Lemmata 5–7, the problem for a variable $X \in Var$ is reduced to check that one of the following two conditions (that are decidable by Propositions 5–6) holds:

- There exists a variable $Y \in Var$ s-reachable from X in $\Re_{\mathrm{SEQ}}$ through a $(K', \emptyset)$-accepting derivation in M_{SEQ} with $K' \subseteq K$, and there exists a derivation $Y \overset{\rho}{\Rightarrow}_{\Re^{K,K_\omega}_{\mathrm{PAR}}}$ such that $\Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR}}}(\rho) = K$ and $\Upsilon^\infty_{M^{K,K_\omega}_{\mathrm{PAR}}}(\rho) \cup \Upsilon^f_{M^{K,K_\omega}_{\mathrm{PAR},\infty}}(\rho) = K_\omega$. Moreover, ρ is either infinite or contains some occurrence of rule in $\Re^{K,K_\omega}_{\mathrm{PAR}} \backslash \Re_{\mathrm{PAR}}$.
- (only when $K = K_\omega$). There exists a (K, K_ω)-accepting infinite derivation in M_{SEQ} from X. □

5. Decidability of the Fairness Problem for unrestricted *MBRS*

In this section we extend the decidability result stated in the previous section to the whole class of *MBRS*, showing that the Fairness Problem for unrestricted *MBRS* is reducible to the Fairness Problem for *MBRS* in normal form. We use a construction very close to that used in [11,12] to solve reachability for *PRS*. We recall that we can assume that the input term in the Fairness Problem is a process variable. Let M be an *MBRS* over *Var* and Σ, with support $\Re$, and with n accepting components. Moreover, let $Var' \supseteq Var$ be a countable set of process variables, and let T_{PAR} be the set of process terms over Var' in which no sequential composition occurs. Now, we describe a procedure that transforms M

into a new *MBRS* M' with the same number of accepting components. Moreover, this procedure has as input also a set of rules $\Re_{\mathrm{AUX}} \subseteq \Re$, and transforms it in $\Re'_{\mathrm{AUX}}$ (the meaning of $\Re_{\mathrm{AUX}}$ is explained below). If M is not in normal form, then there exists some rule r in M that is neither a PAR rule nor a SEQ rule; r can have one of the following forms[7] :

1. $r = t \xrightarrow{a} t_1 \| t_2$ (resp., $r = t_1 \| t_2 \xrightarrow{a} t$) where $\{t, t_1, t_2\} \not\subseteq T_{\mathrm{PAR}}$. Let Z_1, Z_2, Z be fresh variables. We get M' replacing r with the rules $r' = t \rightarrow Z$, $r_3 = Z \rightarrow Z_1 \| Z_2$, $r_1 = Z_1 \rightarrow t_1$, and $r_2 = Z_2 \rightarrow t_2$ (resp., $r_1 = t_1 \rightarrow Z_1$, $r_2 = t_2 \rightarrow Z_2$, $r_3 = Z_1 \| Z_2 \rightarrow Z$, and $r' = Z \rightarrow t$) such that $\Upsilon^f_{M'}(r') = \Upsilon^f_M(r)$ and $\Upsilon^f_{M'}(r_1) = \Upsilon^f_{M'}(r_2) = \Upsilon^f_{M'}(r_3) = \emptyset$.[8] If $r \in \Re_{\mathrm{AUX}}$, then $\Re'_{\mathrm{AUX}} = (\Re_{\mathrm{AUX}} \backslash \{r\}) \cup \{r', r_1, r_2, r_3\}$, otherwise, $\Re'_{\mathrm{AUX}} = \Re_{\mathrm{AUX}}$.
2. $r = t \xrightarrow{a} t_1.t_2$ (resp., $r = t_1.t_2 \xrightarrow{a} t$) where t_2 is not a single variable. Let Z be a fresh variable. We get M' and $\Re'_{\mathrm{AUX}}$ in two steps. First, we substitute Z for t_2 in (left-hand and right-hand sides of) all the rules of M and $\Re_{\mathrm{AUX}}$. Then, we add the rules $r_1 = Z \rightarrow t_2$ and $r_2 = t_2 \rightarrow Z$ such that $\Upsilon^f_{M'}(r_1) = \Upsilon^f_{M'}(r_2) = \emptyset$.
3. $r = t_1 \xrightarrow{a} t_2.X$ (resp., $r = t_2.X \xrightarrow{a} t_1$) where $\{t_1, t_2\} \not\subseteq Var'$. Let Z_1, Z_2 be fresh variables. We get M' replacing r with the rules $r' = t_1 \rightarrow Z_1$, $r_1 = Z_1 \rightarrow Z_2.X$, and $r_2 = Z_2 \rightarrow t_2$ (resp., $r_2 = t_2 \rightarrow Z_2$, $r_1 = Z_2.X \rightarrow Z_1$, and $r' = Z_1 \rightarrow t_1$) such that $\Upsilon^f_{M'}(r') = \Upsilon^f_M(r)$ and $\Upsilon^f_{M'}(r_1) = \Upsilon^f_{M'}(r_2) = \emptyset$. If $r \in \Re_{\mathrm{AUX}}$, then $\Re'_{\mathrm{AUX}} = (\Re_{\mathrm{AUX}} \backslash \{r\}) \cup \{r', r_1, r_2\}$, otherwise, $\Re'_{\mathrm{AUX}} = \Re_{\mathrm{AUX}}$.

The procedure described above preserves the behaviour of the system with respect to the fulfillment of acceptance properties. The set of rules $\Re'_{\mathrm{AUX}}$ is used to keep track of the rules r_1 and r_2 introduced in step 3 which can generate an infinite derivation in M' that may not be simulable in M. After a finite number of applications of this procedure, starting from $\Re_{\mathrm{AUX}} = \emptyset$, we obtain an *MBRS* M' in normal form and a set of rules $\Re'_{\mathrm{AUX}}$. Let $M' = \langle \Re', \langle \Re'_1, \ldots, \Re'_n \rangle \rangle$. Now, let us consider the *MBRS* in normal form with $n+1$ accepting components given by $M_F = \langle \Re', \langle \Re'_1, \ldots, \Re'_n, \Re' \backslash \Re'_{\mathrm{AUX}} \rangle \rangle$. We can prove that, given a variable $X \in Var$ and two sets $K, K_\omega \in P_n$, there exists a (K, K_ω)-accepting infinite derivation in M from X *iff* there exists a (K, K_ω)-accepting infinite derivation in M' from X containing infinite occurrences of rules in $\Re' \backslash \Re'_{\mathrm{AUX}}$ *iff* there exists a $(K \cup \{n+1\}, K_\omega \cup \{n+1\})$-accepting infinite derivation in M_F from X.

6. Complexity issues

We conclude with some considerations about the complexity of the considered problem. Model-checking parallel *PRS* (that are equivalent to Petri nets) w.r.t. the considered *ALTL* fragment, interpreted on infinite runs, is *EXPSPACE*-complete (also for a fixed formula) [10]. *ALTL* model-checking for sequential *PRS* (that are equivalent to Pushdown processes) is less hard, since it is *EXPTIME*-complete [1]. Therefore, model-checking the whole class of *PRS* w.r.t. the considered *ALTL* fragment (restricted to infinite runs) is at least *EXPSPACE*-hard. We have reduced this problem (in polynomial time) to the Fairness Problem (see Theorem 1). Moreover, as seen in Section 5, we can limit ourselves (by a polynomial-time reduction) to consider only *MBRS* in normal form. The algorithm presented in Section 4 to solve the Fairness Problem for *MBRS* in normal form is an exponential reduction (in the number n of accepting components) to the *ALTL* model-checking problem for Petri nets and Pushdown processes: we have to solve an exponential number in n of instances of decision problems about acceptance properties of derivations of parallel and sequential *MBRS*, whose sizes are exponential in n.[9] These last problems (see Propositions 4–6) are polynomial-time reducible to the *ALTL* model-checking problem for Petri nets and Pushdown processes. It was shown [7] that for Petri nets, and for a fixed *ALTL* formula, model checking has the same complexity as reachability (that is *EXPSPACE*-hard, but the best known upper bound is not primitive recursive). Therefore, for n fixed (i.e., for a fixed formula of our *ALTL* fragment) the upper bound given by our algorithm is the same as reachability for Petri nets.

7. Conclusion

We have proved decidability of the model checking problem of unrestricted *PRS* with respect to a meaningful fragment of *ALTL* which, in particular, allows to express boolean combinations of fairness constraints. The result extends the known model checking properties for *PRS* from only simple reachability (shown by Richard Mayr in the first

---

[7] We assume that sequential composition is left-associative. So, when we write $t_1.t_2$, then t_2 is either a single variable or a parallel composition of process terms.

[8] Note that we have not specified the label of the new rules, since it is not relevant.

[9] Note that the number of new rules added in order to built M_{PAR}, M_{SEQ}, $M^{K,K_\omega}_{\mathrm{PAR}}$, and $M^{K,K_\omega}_{\mathrm{PAR},\infty}$ is exponential in n and polynomial in $|Var(\Re)|$.

paper on *PRS*). Actually, we are working on the extension of this result to the linear-time Lamport logic that is the fragment of *ALTL* that uses only the "eventually" and "always" temporal operators (nested arbitrarily). This fragment, suitable encoded in the framework of *MBRS*, corresponds to positive boolean combinations of (generalized) acceptance properties (investigated in the current work) and finite ordering properties. Finite ordering properties require that a rule sequence can be decomposed into a finite number of contiguous pieces such that each of these subsequences contain only occurrences of rules belonging to assigned accepting components of the *MBRS*. It is important to observe that if we add the next operator to the Lamport logic, then by a result of Bouajjani et al. on PA processes [2], the model-checking problem becomes undecidable.

Acknowledgments

I thank Aldo De Luca, Salvatore La Torre, Margherita Napoli, and Adriano Peron for very helpful discussions.

Appendix A

A.1. Proof of Lemma 5

We need the following definitions.

Definition 9. For $t \in T$, the set of *subterms* of t, denoted by $SubTerms(t)$, is defined inductively as follows [10] (where $t_1, t_2 \in T\setminus\{\varepsilon\}$ and $X \in Var \cup \{Z_F\}$):

- $SubTerms(\varepsilon) = \{\varepsilon\}$ and $SubTerms(X) = \{X\}$.
- $SubTerms(t_1.X) = SubTerms(t_1) \cup \{t_1.X\}$.
- $SubTerms(t_1 \| t_2) = \bigcup_{(t'_1,t'_2)\in S} (SubTerms(t'_1) \cup SubTerms(t'_2)) \cup \{t_1 \| t_2\}$, with $S = \{(t'_1, t'_2) \in T \times T \mid t'_1, t'_2 \neq \varepsilon$ and $t_1 \| t_2 = t'_1 \| t'_2\}$. [11]

Definition 10. For $t \in T$, the set of terms $SEQ(t)$ is the subset of $T_{\mathrm{SEQ}}\setminus\{\varepsilon\}$ defined inductively as follows (where $t_1, t_2 \in T\setminus\{\varepsilon\}$ and $X \in Var \cup \{Z_F\}$):

- $SEQ(\varepsilon) = \emptyset$ and $SEQ(X) = \{X\}$.
- $SEQ(t_1.X) = \{t'_1.X \mid t'_1 \in SEQ(t_1)\}$ and $SEQ(t_1 \| t_2) = SEQ(t_1) \cup SEQ(t_2)$.

Definition 11. Let $t \overset{r}{\Rightarrow} t'$ be a single-step derivation in $\Re$ with $t \in T$. We say that r is *applicable at level* 0 in $t \overset{r}{\Rightarrow} t'$, if $t = \bar{t} \| s$, $t' = \bar{t} \| s'$ (for some $\bar{t}, s, s' \in T$), and $r = s \overset{a}{\rightarrow} s'$, for some $a \in \Sigma$.

We say that r is *applicable at level* $k > 0$ *in* $t \overset{r}{\Rightarrow} t'$, if $t = \bar{t} \| (s.X)$, $t' = \bar{t} \| (s'.X)$ (for some $\bar{t}, s, s' \in T$), $s \overset{r}{\Rightarrow} s'$, and r is applicable at level $k-1$ in $s \overset{r}{\Rightarrow} s'$.

The *level of application* of r in $t \overset{r}{\Rightarrow} t'$ is the greatest level of applicability of r in $t \overset{r}{\Rightarrow} t'$.

For a derivation $d = (t \overset{\sigma}{\Rightarrow})$, the level of application of a rule occurrence of σ in d is the level of application of this occurrence in the associated (w.r.t. d) single-step derivation.

Proof of Lemma 5. ($\Rightarrow$) Let $d = (X \overset{\sigma}{\Rightarrow})$ be a (K, K_ω)-accepting infinite derivation in M from variable X. We can assume that d does *not* belong to $\Pi(K, K_\omega)$ (otherwise, the assertion is trivial). Then, d can be written in the form

$$X \overset{\sigma_1}{\Rightarrow} t \| Z \overset{r}{\Rightarrow} t \| (Z'.W) \overset{\sigma_2}{\Rightarrow}, \tag{7}$$

[10] We recall that T denotes the set of terms *in normal form* over $Var \cup \{Z_F\}$.

[11] We recall that we identify terms with their equivalence classes. In particular, $t_1 = t_2$ (resp., $t_1 \neq t_2$) is used to mean that t_1 is equivalent (resp., not equivalent) to t_2.

where $r = Z \xrightarrow{a} Z'.W$, and there exists a subderivation $d' = (Z' \overset{\sigma_2'}{\Rightarrow})$ of $t \| (Z'.W) \overset{\sigma_2}{\Rightarrow}$ from Z' that is a (K, K_ω)-accepting infinite derivation in M from Z'. Moreover, by definition of subderivation there is a derivation of the form

$$t \overset{\sigma_2 \backslash \sigma_2'}{\Rightarrow}. \tag{8}$$

First, assume that $K \neq K_\omega$. Then, $K \supset K_\omega$ and $K \backslash K_\omega = \{i \in \{1, \ldots, n\} | \ \sigma$ contains a *finite non-null* number of occurrences of rules in $\Re_i\}$. Therefore, for each $i \in K \backslash K_\omega$, it is defined the greatest level of application, denoted by $h_i(d)$, of occurrences of rules of $\Re_i$ in the derivation d. The proof is by induction on $\max_{i \in K \backslash K_\omega}\{h_i(d)\}$. Note that $\max_{i \in K \backslash K_\omega}\{h_i(d)\} = 0$ implies that d belongs to $\Pi(K, K_\omega)$. Indeed in this case, since for any subderivation d'' of d the level of application of an occurrence of rule of d'' in the derivation d is not null, we deduce that the subderivations of d do not contain occurrences of rules in $\Re_i$ for any $i \in K \backslash K_\omega$. Therefore, assume that $\max_{i \in K \backslash K_\omega}\{h_i(d)\} > 0$. By the definition of subderivation and Definition 11, it follows that $\max_{i \in K \backslash K_\omega}\{h_i(d')\} < \max_{i \in K \backslash K_\omega}\{h_i(d)\}$. Then, by the induction hypothesis, condition 1 (in the enunciation) is satisfied for variable Z'. Since $r = Z \xrightarrow{a} Z'.W \in \Re_{\text{SEQ}}$ and $\Upsilon_M^f(r) = \Upsilon_{M_{\text{SEQ}}}^f(r) \subseteq K$, it is sufficient to prove that Z is s-reachable from X in $\Re_{\text{SEQ}}$ through a $(K', \emptyset)$-accepting derivation (in M_{SEQ}) with $K' \subseteq K$. By Lemma 2, applied to the derivation $X \overset{\sigma_1}{\Rightarrow} t \| Z$ in (7), it follows that $X \overset{\rho_1}{\Rightarrow}_{\Re_{\text{PAR}}} p \| Z$ for some $p \in T_{\text{PAR}}$ and $\Upsilon_{M_{\text{PAR}}}^f(\rho_1) = \Upsilon_M^f(\sigma_1) \subseteq K$. By the definition of M_{SEQ} we obtain the assertion.

Now, assume that $K = K_\omega$. For $|K| = 0$, the proof is simple and is similar to the last part of the previous case. For $|K| > 0$, it is sufficient to prove the following for all $i \in K$:

A Either condition 1 (in the enunciation) is satisfied, or there exists a variable $Y \in Var$ such that Y is s-reachable from X in M_{SEQ} through a $(K_i, \emptyset)$-accepting derivation (in M_{SEQ}) with $\{i\} \subseteq K_i \subseteq K$, and there exists a (K, K)-accepting infinite derivation in M from variable Y.

We prove Property **A** by induction on the level of application h of the first occurrence of rules of $\Re_i$ in the derivation d. If $h = 0$, then by (7) we deduce that the rule sequence $\sigma_1 r(\sigma_2 \backslash \sigma_2')$ must contain this occurrence of rule of $\Re_i$. Then, by (7) and (8) there exists a derivation of the form $X \overset{\sigma_1}{\Rightarrow} t \| Z \overset{\lambda}{\Rightarrow} t' \| Z \overset{r}{\Rightarrow} t' \| (Z'.W)$ such that $\{i\} \subseteq \Upsilon_M^f(\sigma_1 \lambda r) \subseteq K$ (where λ is a prefix of $\sigma_2 \backslash \sigma_2'$). By Lemma 2, applied to the derivation $X \overset{\sigma_1 \lambda}{\Rightarrow} t' \| Z$, it follows that $X \overset{\rho}{\Rightarrow}_{\Re_{\text{PAR}}} p \| Z$ for some $p \in T_{\text{PAR}}$ and $\Upsilon_{M_{\text{PAR}}}^f(\rho) = \Upsilon_M^f(\sigma_1 \lambda)$. By the definition of M_{SEQ} we obtain that $X \overset{\mu}{\Rightarrow}_{\Re_{\text{SEQ}}} Z \overset{r}{\Rightarrow}_{\Re_{\text{SEQ}}} Z'.W$, with $\Upsilon_{M_{\text{SEQ}}}^f(\mu) = \Upsilon_{M_{\text{PAR}}}^f(\rho)$ and $\Upsilon_{M_{\text{SEQ}}}^f(r) = \Upsilon_M^f(r)$. Then, $\Upsilon_{M_{\text{SEQ}}}^f(\mu r) = \Upsilon_M^f(\sigma_1 \lambda r)$ and, in particular, $\{i\} \subseteq \Upsilon_{M_{\text{SEQ}}}^f(\mu r) \subseteq K$. Since d' is a (K, K)-accepting infinite derivation in M from Z', Property **A** holds. Now, assume that $h > 0$. If the rule sequence $\sigma_1 r(\sigma_2 \backslash \sigma_2')$ contains some occurrence of rule of $\Re_i$, then we proceed as in the base step. Otherwise, σ_2' contains the first occurrence of rules of $\Re_i$ in σ. Clearly, this occurrence is the first occurrence of rules of $\Re_i$ in σ_2' (since σ_2' is a subsequence of σ), and is applied at level $h' < h$ in the subderivation $d' = (Z' \overset{\sigma_2'}{\Rightarrow}_{\Re})$. By the induction hypothesis, Property **A** holds for variable Z'. Then, it is sufficient to prove that Z' is s-reachable from X in M_{SEQ} through a $(K', \emptyset)$-accepting derivation in M_{SEQ} with $K' \subseteq K$. As before, applying Lemma 2 to the derivation $X \overset{\sigma_1}{\Rightarrow}_{\Re} t \| Z$ in (7), by the definition of M_{SEQ} we obtain the assertion.

($\Leftarrow$) First, we prove the following property:

B Let $t, t' \in T_{\text{SEQ}}$ and $s \in T$ such that $t \in SEQ(s)$. If $t \overset{r}{\Rightarrow}_{\Re_{\text{SEQ}}} t'$ with $r \in \Re_{\text{SEQ}}$, then there exists $s' \in T$ such that $t' \in SEQ(s')$, $s \overset{\sigma}{\Rightarrow}_{\Re} s'$, $\Upsilon_M^f(\sigma) = \Upsilon_{M_{\text{SEQ}}}^f(r)$, and $|\sigma| > 0$.

By definition of M_{SEQ} there are two cases:

- $r = Y \xrightarrow{a} Z_1.Z_2 \in \Re$, $t = Y.t_1$ and $t' = Z_1.Z_2.t_1$ for some $t_1 \in T_{\text{SEQ}}$. Since $t \in SEQ(s)$ (note that Y is a subterm of s), it easily follows (by induction on the structure of s) that there is $s' \in T$ such that $t' \in SEQ(s')$ and $s \overset{r}{\Rightarrow}_{\Re} s'$. Since $\Upsilon_M^f(r) = \Upsilon_{M_{\text{SEQ}}}^f(r)$, property **B** is satisfied.
- $r = Y \xrightarrow{K'} Z$ with $Y, Z \in Var$. By definition of M_{SEQ}, there is a derivation in M_{PAR} of the form $Y \overset{\sigma}{\Rightarrow}_{\Re_{\text{PAR}}} p \| Z$ for some $p \in T_{\text{PAR}}$, such that $\Upsilon_{M_{\text{PAR}}}^f(\sigma) = \Upsilon_{M_{\text{SEQ}}}^f(r)$ and $|\sigma| > 0$. By Lemma 3 there is a term st such that $Y \overset{\rho}{\Rightarrow}_{\Re} st \| Z$, $\Upsilon_M^f(\rho) = \Upsilon_{M_{\text{PAR}}}^f(\sigma)$, and $|\rho| > 0$. Hence, $\Upsilon_M^f(\rho) = \Upsilon_{M_{\text{SEQ}}}^f(r)$. Now, $t = Y.t_1$ and $t' = Z.t_1$ for some $t_1 \in T_{\text{SEQ}}$.

Since $t \in SEQ(s)$ and $Y \stackrel{\rho}{\Rightarrow}_{\Re} st\|Z$, it easily follows (by induction on the structure of s) that there is $s' \in T$ such that $t' \in SEQ(s')$ and $s \stackrel{\rho}{\Rightarrow}_{\Re} s'$. Hence, Property **B** holds.

Now, we can prove the assertion. If condition 2 (in the enunciation) is satisfied, then, since $X \in SEQ(X)$, the thesis easily follows from Property **B** above. Assume that condition 1 holds instead. Therefore, we have that $X \stackrel{\rho}{\Rightarrow}_{\Re_{\text{SEQ}}} t$ with $t = Y.t'$ (for some $t' \in T_{\text{SEQ}}$) and $\Upsilon^f_{M_{\text{SEQ}}}(\rho) \subseteq K$, and there exists a (K, K_ω)-accepting infinite derivation in M of the form $Y \stackrel{\sigma}{\Rightarrow}_{\Re}$. Since $X \in SEQ(X)$, by Property **B** it follows that there exists $s \in T$ such that $t \in SEQ(s)$ and $X \stackrel{\eta}{\Rightarrow}_{\Re} s$ with $\Upsilon^f_M(\eta) \subseteq K$. Since $Y \in SubTerms(s)$ and $Y \stackrel{\sigma}{\Rightarrow}_{\Re}$, we deduce that $s \stackrel{\sigma}{\Rightarrow}_{\Re}$. Therefore, there exists a derivation of the form $X \stackrel{\eta}{\Rightarrow}_{\Re} s \stackrel{\sigma}{\Rightarrow}_{\Re}$, which is a (K, K_ω)-accepting infinite derivation in M from X. This concludes the proof. □

A.2. Proof of Lemma 7

In order to prove Lemma 7, we use a mapping for encoding pairs of integers by single integers. In particular, we consider the following bijective mapping from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ [6]

$$\langle\,\rangle: (x, y) \in \mathbb{N} \times \mathbb{N} \to 2^x(2y + 1) - 1$$

Let ℓ (resp. $\wp$) be the first (resp., second) component of $< >^{-1}$. Then,
1. for all $z \in \mathbb{N}$, $\ell(z), \wp(z) \leqslant z$,
2. for all $z, z' \in \mathbb{N}$, if $z > z'$ and $\ell(z) = \ell(z')$, then $\wp(z) > \wp(z')$.

Now, we introduce a new function $next : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ defined as follows:

$$next(x, 0) = (x, 0),$$

$$next(x, y + 1) = \begin{cases} (\ell(y), \wp(y) + 1) & \text{if } next(x, y) = (\ell(y), \wp(y)), \\ next(x, y) & \text{otherwise.} \end{cases}$$

The following lemma establishes some properties of *next*. The proof is simple.

Lemma 8.
1. *For all* $x, y \in \mathbb{N}$, *if* $y \leqslant x$, *then* $next(x, y) = (x, 0)$.
2. *For all* $x \in \mathbb{N}$, $next(\ell(x), x) = (\ell(x), \wp(x))$.
3. *For all* $x, i \in \mathbb{N}$, *if* $i \neq \ell(x)$, *then* $next(i, x + 1) = next(i, x)$.

Now, we can prove Lemma 7. Let $p \stackrel{\sigma}{\Rightarrow}_{\Re^{K,K_\omega}_{\text{PAR}}}$ with $p \in T_{\text{PAR}}$ such that σ is either infinite or contains some occurrence of rule in $\Re^{K,K_\omega}_{\text{PAR}} \backslash \Re_{\text{PAR}}$. Let λ be the subsequence of σ containing all, and only, the occurrences of rules in $\Re^{K,K_\omega}_{\text{PAR}} \backslash \Re_{\text{PAR}}$. Assume that λ is infinite. The proof for λ finite (and possibly empty) is simpler. Now, $\lambda = r_0 r_1 r_2 \ldots$, where for all $h \in \mathbb{N}$, $r_h \in \Re^{K,K_\omega}_{\text{PAR}} \backslash \Re_{\text{PAR}}$, and σ can be written in the form $\rho_0 r_0 \rho_1 r_1 \rho_2 r_2 \ldots$, where $\sigma \backslash \lambda = \rho_0 \rho_1 \rho_2 \ldots$ and for all $h \in \mathbb{N}$, ρ_h is a finite rule sequence (possibly empty) in $\Re_{\text{PAR}}$. For all $h \in \mathbb{N}$, we denote by σ^h the suffix of σ given by $\rho_h r_h \rho_{h+1} r_{h+1} \ldots$. Now, we prove that there exist a sequence of terms in T_{PAR}, $(p_h)_{h \in \mathbb{N}}$, a sequence of variables $(X_h)_{h \in \mathbb{N}}$ (in *Var*), and a sequence of terms $(t_h)_{h \in \mathbb{N}}$ such that for all $h \in \mathbb{N}$:

(i) $p_0 = p$,

(ii) $p_h \stackrel{\sigma^h}{\Rightarrow}_{\Re^{K,K_\omega}_{\text{PAR}}}$,

(iii) $p_h \stackrel{\eta_h}{\Rightarrow}_{\Re} p_{h+1} \| t_h \| X_h$ with $\Upsilon^f_M(\eta_h) = \Upsilon^f_{M^{K,K_\omega}_{\text{PAR}}}(\rho_h)$,

(iv) $X_h \stackrel{\pi_h}{\Rightarrow}_{\Re}$ with π_h infinite, $\Upsilon^f_M(\pi_h) = \Upsilon^f_{M^{K,K_\omega}_{\text{PAR}}}(r_h)$ and $\Upsilon^\infty_M(\pi_h) = \Upsilon^f_{M^{K,K_\omega}_{\text{PAR},\infty}}(r_h)$.

Assume that $p_h \stackrel{\sigma^h}{\Rightarrow}_{\Re^{K,K_\omega}_{\text{PAR}}}$ (for $h = 0$, this holds). Then, this derivation can be written as $p_h \stackrel{\rho_h}{\Rightarrow}_{\Re^{K,K_\omega}_{\text{PAR}}} p_{h+1} \| p' \| X_h \stackrel{r_h}{\Rightarrow}_{\Re^{K,K_\omega}_{\text{PAR}}} p_{h+1} \| p' \| Z_F \stackrel{\sigma^{h+1}}{\Rightarrow}_{\Re^{K,K_\omega}_{\text{PAR}}}$ where $r_h = X_h \stackrel{K',K'_\omega}{\rightarrow} Z_F$, $\Upsilon^f_{M^{K,K_\omega}_{\text{PAR}}}(r_h) = K'$, $\Upsilon^f_{M^{K,K_\omega}_{\text{PAR},\infty}}(r_h) = K'_\omega$, p_{h+1} does not contain occurrences of Z_F, and p' does not contain occurrences of variables in *Var*. By the definition of $\Re_{\mathrm{PAR}}^{K,K_\omega}$, there is an infinite derivation of the form $X_h \overset{\pi_h}{\Rightarrow}_{\Re}$ such that $\Upsilon_M^f(\pi_h) = K'$ and $\Upsilon_M^\infty(\pi_h) = K'_\omega$. Since the left-hand side of each rule in $\Re_{\mathrm{PAR}}^{K,K_\omega}$ does not contain occurrences of Z_F, it follows that $p_{h+1} \overset{\sigma^{h+1}}{\Rightarrow}_{\Re_{\mathrm{PAR}}}$. Since ρ_h is a rule sequence in $\Re_{\mathrm{PAR}}$, by Lemma 3, applied to the derivation $p_h \overset{\rho_h}{\Rightarrow}_{\Re_{\mathrm{PAR}}} p_{h+1}\|p'\|X_h$, it follows that $p_h \overset{\eta_h}{\Rightarrow}_{\Re} p_{h+1}\|t_h\|X_h$ for some term t_h and $\Upsilon_M^f(\eta_h) = \Upsilon_{M_{\mathrm{PAR}}}^f(\rho_h)$. By construction $\Upsilon_{M_{\mathrm{PAR}}}^f(\rho_h) = \Upsilon_{M_{\mathrm{PAR}}^{K,K_\omega}}^f(\rho_h)$. Therefore, properties (ii)–(iv) hold for any $h \in \mathbb{N}$.

For all $h \in \mathbb{N}$, the infinite derivation $X_h \overset{\pi_h}{\Rightarrow}_{\Re}$ in (iv) can be written as

$$s_{(h,0)} \overset{r_{(h,0)}}{\Rightarrow}_{\Re} s_{(h,1)} \overset{r_{(h,1)}}{\Rightarrow}_{\Re} s_{(h,2)} \ldots, \tag{9}$$

where $s_{(h,0)} = X_h$ and, for all $k \in \mathbb{N}$, $r_{(h,k)} \in \Re$. Let $\bar{r}_k$ be the rule $r_{(\ell(k),\wp(k))}$, and $s_h(k)$ be the term $s_{\mathrm{next}(h,k)}$. Now, we show that for all $k \in \mathbb{N}$,

$$p_{k+1}\|t_0\|\ldots\|t_k\|s_0(k)\|s_1(k)\|\ldots\|s_k(k) \overset{\eta_{k+1}\bar{r}_k}{\Rightarrow}_{\Re} p_{k+2}\|t_0\|\ldots\|t_k\|t_{k+1}\|s_0(k+1)\|s_1(k+1)\|\ldots\|s_{k+1}(k+1). \tag{10}$$

By Lemma 8, $s_k(k) = s_{\mathrm{next}(k,k)} = s_{(k,0)} = X_k$. So, by Property (iii) it holds that

$$p_{k+1}\|t_0\|\ldots\|t_k\|s_0(k)\|s_1(k)\|\ldots\|s_k(k) \overset{\eta_{k+1}}{\Rightarrow}_{\Re} p_{k+2}\|t_0\|\ldots\|t_k\|t_{k+1}\|s_0(k)\|s_1(k)\|\ldots\|s_k(k)\|s_{k+1}(k+1). \tag{11}$$

Thus, in order to obtain (10) it is sufficient to prove that

$$s_0(k)\|s_1(k)\|\ldots\|s_k(k) \overset{\bar{r}_k}{\Rightarrow}_{\Re} s_0(k+1)\|s_1(k+1)\|\ldots\|s_k(k+1). \tag{12}$$

By Property 2 of Lemma 8, $next(\ell(k),k) = (\ell(k),\wp(k))$ and $next(\ell(k),k+1) = (\ell(k),\wp(k)+1)$. Therefore, $s_{\ell(k)}(k) = s_{(\ell(k),\wp(k))} \overset{\bar{r}_k}{\Rightarrow}_{\Re} s_{(\ell(k),\wp(k)+1)} = s_{\ell(k)}(k+1)$. By Property 3 of Lemma 8, for all $i \neq \ell(k)$, $next(i,k+1) = next(i,k)$. So, for all $i \neq \ell(k)$, $s_i(k+1) = s_i(k)$. Since $\ell(k) \leqslant k$, we obtain evidently (12). Thus, (10) is satisfied for all $k \in \mathbb{N}$. Moreover, since $s_0(0) = X_0$, we have

$$p = p_0 \overset{\eta_0}{\Rightarrow}_{\Re} p_1\|t_0\|s_0(0). \tag{13}$$

Setting $\delta = \eta_0\eta_1\bar{r}_0\eta_2\bar{r}_1\eta_3\bar{r}_2,\ldots$, from (10) and (13) we obtain that $p \overset{\delta}{\Rightarrow}_{\Re}$ with δ infinite. Therefore, it remains to prove that $\Upsilon_M^f(\delta) = \Upsilon_{M_{\mathrm{PAR}}^{K,K_\omega}}^f(\sigma)$ and $\Upsilon_M^\infty(\delta) = \Upsilon_{M_{\mathrm{PAR}}^{K,K_\omega}}^\infty(\sigma) \cup \Upsilon_{M_{\mathrm{PAR},\infty}^{K,K_\omega}}^f(\sigma)$.

Let $\mu = \bar{r}_0\bar{r}_1\bar{r}_2\ldots$. Evidently, $\mu \in Interleave((\pi_h)_{h\in\mathbb{N}})$. By Properties (iii)–(iv), Proposition 3, and the fact that $\sigma = \rho_0 r_0 \rho_1 r_1 \ldots$, we obtain

$$\begin{aligned}\Upsilon_M^f(\delta) &= \bigcup_{h\in\mathbb{N}} \Upsilon_M^f(\eta_h) \cup \Upsilon_M^f(\mu) = \bigcup_{h\in\mathbb{N}} \Upsilon_{M_{\mathrm{PAR}}^{K,K_\omega}}^f(\rho_h) \cup \bigcup_{h\in\mathbb{N}} \Upsilon_M^f(\pi_h)\\ &= \bigcup_{h\in\mathbb{N}} \Upsilon_{M_{\mathrm{PAR}}^{K,K_\omega}}^f(\rho_h) \cup \bigcup_{h\in\mathbb{N}} \Upsilon_{M_{\mathrm{PAR}}^{K,K_\omega}}^f(r_h) = \Upsilon_{M_{\mathrm{PAR}}^{K,K_\omega}}^f(\sigma).\end{aligned}$$

By construction, for all $r \in \Re_{\mathrm{PAR}}$, $\Upsilon_{M_{\mathrm{PAR},\infty}^{K,K_\omega}}^f(r) = \emptyset$. Recalling that $\lambda = r_0r_1r_2\ldots$, by Properties (iii)–(iv) and Proposition 3 we obtain

$$\begin{aligned}\Upsilon_M^\infty(\delta) &= \bigoplus_{h\in\mathbb{N}} \Upsilon_M^f(\eta_h) \cup \Upsilon_M^\infty(\mu) = \bigoplus_{h\in\mathbb{N}} \Upsilon_{M_{\mathrm{PAR}}^{K,K_\omega}}^f(\rho_h) \cup \bigcup_{h\in\mathbb{N}} \Upsilon_M^\infty(\pi_h) \cup \bigoplus_{h\in\mathbb{N}} \Upsilon_M^f(\pi_h)\\ &= \Upsilon_{M_{\mathrm{PAR}}^{K,K_\omega}}^\infty(\sigma\backslash\lambda) \cup \bigcup_{h\in\mathbb{N}} \Upsilon_{M_{\mathrm{PAR},\infty}^{K,K_\omega}}^f(r_h) \cup \bigoplus_{h\in\mathbb{N}} \Upsilon_{M_{\mathrm{PAR}}^{K,K_\omega}}^f(r_h)\\ &= \Upsilon_{M_{\mathrm{PAR}}^{K,K_\omega}}^\infty(\sigma\backslash\lambda) \cup \Upsilon_{M_{\mathrm{PAR},\infty}^{K,K_\omega}}^f(\sigma) \cup \Upsilon_{M_{\mathrm{PAR}}^{K,K_\omega}}^\infty(\lambda) = \Upsilon_{M_{\mathrm{PAR}}^{K,K_\omega}}^\infty(\sigma) \cup \Upsilon_{M_{\mathrm{PAR},\infty}^{K,K_\omega}}^f(\sigma).\end{aligned}$$

This concludes the proof. □

References

[1] A. Bouajjani, J. Esparza, O. Maler, Reachability analysis of pushdown automata: application to model-checking, in: Proc. Eighth Internat. Conf. on Concurrency Theory (CONCUR'97), Lecture Notes in Computer Science, Vol. 1243, Springer, Berlin, 1997, pp. 135–150.
[2] A. Bouajjani, P. Habermehl, Constrained properties, semilinear systems, and Petri nets, in: Proc. Seventh Internat. Conf. on Concurrency Theory (CONCUR'96), Lecture Notes in Computer Science, Vol. 1119, Springer, Berlin, 1996, pp. 481–497.
[3] A. Bouajjani, T. Touili, Reachability analysis of process rewrite systems, in: Proc. 23rd Internat. Conf. on Foundations of Software Technology and Theoretical Computer Science (FSTTCS'03), Lecture Notes in Computer Science, Vol. 2914, Springer, Berlin, 2003, pp. 74–87.
[4] O. Burkart, D. Caucal, F. Moller, B. Steffen, Verification on infinite structures, in: Handbook of Process Algebra, North-Holland, Amsterdam, 2001, pp. 545–623.
[5] O. Burkart, B. Steffen, Pushdown processes: parallel composition and model checking, in: Proc. Fifth Internat. Conf. on Concurrency Theory (CONCUR'94), Lecture Notes in Computer Science, Vol. 836, Springer, Berlin, 1994, pp. 98–113.
[6] M.D. Davis, E.J. Weyuker, Computability, Complexity, and Languages, Academic Press, New York, 1983 pp. 47–49.
[7] J. Esparza, On the decidability of model checking for several μ-calculi and Petri nets, in: Proc. Trees in Algebra and Programming (CAAP'94), Lecture Notes in Computer Science, Vol. 787, Springer, Berlin, 1994, pp. 115–129.
[8] J. Esparza, Decidability of model checking for infinite-state concurrent systems, Acta Informatica 34 (2) (1997) 85–107.
[9] J. Esparza, D. Hansel, P. Rossmanith, S. Schwoon, Efficient algorithms for model checking pushdown systems, in: Proc. 12th Internat. Conf. on Computer Aided Verification (CAV'00), Lecture Notes in Computer Science, Vol. 1855, Springer, Berlin, 2000, pp. 232–247.
[10] P. Habermehl, On the complexity of the linear-time mu-calculus for Petri nets, in: Proc. Internat. Conf. on Application and Theory of Petri Nets, Lecture Notes in Computer Science, Vol. 1248, Springer, Berlin, 1997, pp. 102–116.
[11] R. Mayr, Decidability and complexity of model checking problems for infinite-state systems. Ph.D. Thesis, Technical University of Munich, 1998.
[12] R. Mayr, Process rewrite systems, Inform. Comput. 156 (2000) 264–286.
[13] I. Walukiewic, Pushdown processes: games and model checking, in: Proc. Eighth Internat. Conf. on Computer Aided Verification (CAV'96), Lecture Notes in Computer Science, Vol. 1102, Springer, Berlin, 1996, pp. 62–74.