Complexity and Succinctness Issues for Linear-Time Hybrid Logics

Laura Bozzelli and Ruggero Lanotte

Università dell'Insubria, Via Valleggio 11, 22100 - Como, Italy

Abstract. Full linear-time hybrid logic (HL) is a non-elementary and equally expressive extension of standard LTL + past obtained by adding the well-known binder operators $\downarrow$ and $\exists$. We investigate complexity and succinctness issues for HL in terms of the number of variables and nesting depth of binder modalities. First, we present *direct* automata-theoretic decision procedures for satisfiability and model-checking of HL, which require space of exponential height equal to the nesting depth of binder modalities. The proposed algorithms are proved to be asymptotically optimal by providing matching lower bounds. Second, we show that for the one-variable fragment of HL, the considered problems are elementary and, precisely, EXPSPACE-complete. Finally, we show that for all $0 \leq h < k$, there is a succinctness gap between the fragments HL^k and HL^h with binder nesting depth at most k and h, respectively, of exponential height equal to $k - h$.

1 Introduction

Hybrid logics extend modal and temporal logics with features from first-order logic which provide very natural modeling facilities [BS98]. In particular, they provide a type of atomic formulas, called *nominals*, which represent names for states of a model (hence, nominals correspond to constants in first-order logic). Moreover, they contain the *at operator* $@_n$ which gives 'random' access to the state named by n. They may also include the *downarrow binder operator* $\downarrow x$, which assigns the variable name x to the current state, and the *existential binder operator* $\exists x$, which binds x to some state in the model. Applications of hybrid logics range from verification tasks to reasoning about semistructured data [FR06]. Here, we focus on complexity issues for hybrid logics.

Satisfiability of hybrid logics including the binder operator $\downarrow$ or $\exists$ and interpreted on general structures is undecidable, also for small fragments [ABM01, CF05]. For the class of linear structures (based on the frame of the natural numbers with the usual ordering), the problem is instead decidable [FRS03]. However, satisfiability and model checking of full linear-time hybrid logic (HL, for short), an equally-expressive extension of standard LTL + past (PLTL) [Pnu77] with the binders operators $\downarrow$ and $\exists$, are non-elementarily decidable (recall that for LTL and PLTL, these problems are instead PSPACE-complete [SC85, Var88]), and this already holds for the fragment $\mathsf{HL}(\downarrow)$ of HL obtained by disallowing the $\exists$-operator. This is a consequence of the fact that standard first-order logic over words (FO), which is non-elementary [Sto74], can be linearly translated into $\mathsf{HL}(\downarrow)$ [FRS03]. Moreover, by results in [Sto74], there is a non-elementary gap between the succinctness of $\mathsf{HL}(\downarrow)$ and PLTL. Recently, Schwentick et al. [SW07] show that satisfiability

S. Hölldobler, C. Lutz, and H. Wansing (Eds.): JELIA 2008, LNAI 5293, pp. 48–61, 2008.

of the one-variable fragment $\mathsf{HL}_1(\downarrow)$ of $\mathsf{HL}(\downarrow)$ is elementary and precisely EXPSPACE-complete, while the fragment $\mathsf{HL}_2(\downarrow)$ of $\mathsf{HL}(\downarrow)$ using at most two-variables remains non-elementarily decidable.

Our Contribution. In this paper we further investigate the linear-time hybrid logic HL, and focus on complexity and succinctness issues in terms of the number of variables and the nesting depth of binder modalities. Note that as shown in [FRS03], for the linear-time setting, *nominals* and the *at operator* $@_n$ can be linearly translated into PLTL (without using $\exists$ and $\downarrow$). Thus, they are not considered in this paper. For each $h \geq 1$, let HL^h and $\mathsf{HL}^h(\downarrow)$ be the fragments of HL and $\mathsf{HL}(\downarrow)$, respectively, consisting of formulas with nesting depth of the binder operators at most h, and let h-EXPSPACE be the class of languages which can be decided in space of exponential height h.

First, we present automata-theoretic decision procedures for satisfiability and model checking of HL based on a translation of HL formulas into a subclass of generalized Büchi alternating word automata (AWA). The construction is *direct* and compositional and is based on a characterization of the satisfaction relation for a given formula φ, in terms of sequences of sets associated with φ (which generalize the classical notion of Hintikka-set of LTL) satisfying determined requirements which can be checked by AWA. The proposed translation lead to algorithms which run in space of exponential height equal to the nesting depth of binder modalities. As a consequence for each $h \geq 1$, satisfiability and model checking of HL^h and $\mathsf{HL}^h(\downarrow)$ are in h-EXPSPACE. We show that the proposed algorithms are asymptotically optimal by providing matching lower bounds, and in particular, we show that h-EXPSPACE-hardness already holds for the fragment $\mathsf{HL}_2^h(\downarrow)$ of $\mathsf{HL}^h(\downarrow)$ using at most two variables.

Second, we show that the complexity of satisfiability and model checking for the one-variable fragment HL_1 of HL is elementary and, precisely, EXPSPACE-complete. Note that our result for satisfiability does not follow from EXPSPACE-completeness of the same problem for the one-variable fragment $\mathsf{HL}_1(\downarrow)$ of $\mathsf{HL}(\downarrow)$ [SW07]. In fact, as shown in [FRS03], the $\exists$-operator can be linearly translated into $\mathsf{HL}(\downarrow)$, but the resulting formula contains an additional variable. In particular, HL_1 can be linearly translated into $\mathsf{HL}_2(\downarrow)$ (using two variables), which is already non-elementary. Thus, actually, we do not know whether HL_1 can be translated into $\mathsf{HL}_1(\downarrow)$ with an elementary blow-up.

Finally, we show that for all $0 \leq k < h$, there is a succinctness gap between HL^h and HL^k of exponential height equal to $h - k$.

Remark. Recall that the only known automata-theoretic decision procedures for FO, where the starting point is the work of Büchi [Buc62] on the decidability of MSO over infinite words (and its first-order fragment FO), are not direct and are based on the closure of ω-regular languages under projection and boolean operations. However, complementation for Büchi nondeterministic automata (NWA) is not trivial and the known constructions such as that based on Safra's determinization result [Saf88] are quite complicated. On the other hand, if we use alternating automata, then complementation (by dualization) is easy, but projection, which is trivial for Büchi NWA, is hard and effectively requires translating AWA back to NWA. Thus, the novelty of the proposed automata-theoretic approach for HL, which can be used also for FO (since FO is linearly translatable in HL) is that like the standard automatic-theoretic approach for LTL, it is based on a direct construction which does not use closure results.

Due to lack of space, many proofs are omitted and can be found in [BL08].

2 Preliminaries

Definition 1. Let $\mathbb{N}$ be the set of natural numbers. For all $n, h \in \mathbb{N}$, let $\mathsf{Tower}(n,h)$ be defined as: $\mathsf{Tower}(n,0) = n$ and $\mathsf{Tower}(n,h+1) = 2^{\mathsf{Tower}(n,h)}$. For each $h \geq 0$, $\exp[h]$ denotes the class of functions $f : \mathbb{N} \to \mathbb{N}$ such that for some constant $c \geq 1$, $f(n) = \mathsf{Tower}(n^c, h)$ for each n. We denote by h-EXPSPACE the class of languages decided by $\exp[h]$-space bounded deterministic Turing machines.

2.1 The Linear-Time Hybrid Logic HL

For a finite alphabet Σ and a finite or infinite word $w = \sigma_0\sigma_1\ldots$ over Σ, $|w|$ denotes the length of w (we set $w = \infty$ if w is infinite). For each $0 \leq i < |w|$, $w(i)$ denotes the i^{th} symbol σ_i of w, w^i denotes the i^{th} suffix of w, i.e. the word $w^i = \sigma_i\sigma_{i+1}\ldots$, and for $0 \leq i \leq j < |w|$, $w[i,j]$ denotes the finite word $w[i,j] = \sigma_i\sigma_{i+1}\ldots\sigma_j$.

Fix a countable set $\{x_1, x_2, \ldots\}$ of (position) variables. The set of HL formulas over a finite set AP of atomic propositions is defined by the following syntax:

$$\varphi := \top \mid p \mid x_h \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{X}^{dir}\varphi \mid \varphi\,\mathsf{U}^{dir}\varphi \mid \exists x_h.\varphi$$

where $\top$ denotes `true`, $p \in AP$, $dir \in \{+,-\}$, X^+ and U^+ are the future temporal operators "forward next" and "forward until", X^- ("backward next") and U^- ("backward until") are their past counterparts, and $\exists$ is the *existential binder* operator. We also use classical shortcuts: $\mathsf{F}^+\varphi := \top\,\mathsf{U}^+\varphi$ ("forward eventually") and $\mathsf{F}^-\varphi := \top\,\mathsf{U}^-\varphi$ ("backward eventually"), and their duals $\mathsf{G}^+\varphi := \neg\mathsf{F}^+\neg\varphi$ ("forward always") and $\mathsf{G}^-\varphi := \neg\mathsf{F}^-\neg\varphi$ ("backward always"). Moreover, the *downarrow binder* operator $\downarrow$ [Gor96] can be introduced as an abbreviation as follows: $\downarrow x_h.\varphi := \exists x_h.(x_h \wedge \varphi)$.

The notion of *free* variable (w.r.t. the binder modalities) are obvious generalizations from first-order logic. A formula φ is *open* if there is some variable which occurs free in φ. A non-open formula is called *sentence*. HL over AP is interpreted on finite or infinite words w over 2^{AP}. A *valuation* for w is a mapping g assigning to each variable a position $j < |w|$ of w. The satisfaction relation $(w,i,g) \models \varphi$, meaning that φ holds at position i along w w.r.t. the valuation g, is inductively defined as follows (we omit the rules for propositions in AP, boolean connectives, and the past temporal operators):

$$\begin{array}{lll}
(w,i,g) \models x_h & \texttt{iff} & i = g(x_h) \\
(w,i,g) \models \mathsf{X}^+\varphi & \texttt{iff} & i+1 < |w| \text{ and } (w,i+1,g) \models \varphi \\
(w,i,g) \models \varphi_1\,\mathsf{U}^+\varphi_2 & \texttt{iff} & \text{there is } i \leq n < |w|.\,(w,n,g) \models \varphi_2 \text{ and} \\
 & & \quad\text{for all } i \leq k < n.\,(w,k,g) \models \varphi_1 \\
(w,i,g) \models \exists x_h.\varphi & \texttt{iff} & (w,i,g[x_h \leftarrow m]) \models \varphi \text{ for some } m < |w|
\end{array}$$

where $g[x_h \leftarrow m](x_h) = m$ and $g[x_h \leftarrow m](x_i) = g(x_i)$ for $i \neq h$. Thus, the $\exists x$-operator binds the variable x to some position in the given word, while the $\downarrow x$-operator binds the variable x to the current position. Note that the satisfaction relation depends only on the values assigned to the variables occurring free in the given formula φ. We write $(w,i) \models \varphi$ to mean that $(w,i,g_0) \models \varphi$, where g_0 maps each variable to position 0.

In the following, unless stated otherwise, a given HL formula is assumed to be a sentence. The size $|\varphi|$ of a HL formula φ is the number of distinct subformulas of φ. Note that the fragment of HL obtained by disallowing variables and the binders operators corresponds to standard LTL + past (PLTL) [Pnu77]. We denote by HL($\downarrow$), the HL fragment given by PLTL + variables + $\downarrow$-operator. W.l.o.g. we assume that if a formula φ uses at most n-variables, these variables are $x_1,\ldots,x_n$, and we write $(w,i,j_1,\ldots,j_n) \models \varphi$ to mean that $(w,i,g) \models \varphi$ for any valuation for w assigning to variable x_h the value j_h for each $1 \leq h \leq n$. For each $k \geq 0$, HL_k (resp., $\mathsf{HL}_k(\downarrow)$) denotes the fragment of HL (resp., HL($\downarrow$)) using at most k variables. For a HL (resp., HL($\downarrow$)) formula φ, $\mathsf{d}_\exists(\varphi)$ (resp., $\mathsf{d}_\downarrow(\varphi)$) denotes the nesting depth of modality $\exists$ (resp., $\downarrow$) in φ. For all $h,k \geq 0$, HL^h and HL^h_k denote the fragments of HL and HL_k, respectively, where the nesting depth of the $\exists$-operator is at most h. The fragments $\mathsf{HL}^h(\downarrow)$ and $\mathsf{HL}^h_k(\downarrow)$ can be defined similarly. Note that since PLTL and HL are equally expressive [FRS03], the mentioned HL fragments, which extend PLTL, are equally expressive.

Global and Initial Equivalence. Two HL formulas φ_1 and φ_2 are said to be (*globally*) *equivalent* if for each word w and $i < |w|$, $(w,i) \models \varphi_1$ iff $(w,i) \models \varphi_2$. Moreover, φ_1 and φ_2 are said to be *initially equivalent* if for each non-empty word w, $(w,0) \models \varphi_1$ iff $(w,0) \models \varphi_2$. Note that (global) equivalence implies initial equivalence.

Decision problems. A *Kripke structure* over AP is a tuple $\mathcal{K} = \langle S, s_0, \Delta, L \rangle$, where S is a finite set of states, $s_0 \in S$ is an initial state, $\Delta \subseteq S \times S$ is a transition relation that must be total, and $L : S \rightarrow 2^{AP}$ maps each state s to the set of propositions that hold in s. A path of $\mathcal{K}$ is an infinite sequence $\pi = s_0 s_1 \ldots$ such that $(s_i, s_{i+1}) \in \Delta$ for each $i \geq 0$.

We are interested in the following decision problems for a given linear-time hybrid logic $\mathfrak{F}$ over AP (such as HL or one of its mentioned fragments), where for a $\mathfrak{F}$-formula φ, $\mathcal{L}(\varphi)$ denotes the set of *infinite* words w over 2^{AP} such that $(w,0) \models \varphi$:

- The *satisfiability problem* is to decide given a formula φ of $\mathfrak{F}$, whether $\mathcal{L}(\varphi) \neq \emptyset$;
- The (*finite-state*) *model checking problem* is to decide given a formula φ of $\mathfrak{F}$ and a Kripke structure $\mathcal{K}$ over AP, whether $\mathcal{K}$ *satisfies* φ, i.e., whether for all paths $\pi = s_0 s_1 \ldots$ of $\mathcal{K}$, condition $(L(s_0)L(s_1)\ldots,0) \models \varphi$ holds.

Note that the $\exists$-operator can be expressed in terms of the $\downarrow$-operator as: $\exists x.\varphi \equiv \downarrow y.\mathsf{E}\downarrow x.\mathsf{E}(y \wedge \varphi)$, where $\mathsf{E}\psi := \mathsf{F}^-(\neg \mathsf{X}^- \top \wedge \mathsf{F}^+\psi)$. The use of an additional variable y seems necessary, and actually, we do not know whether HL_1 can be translated into $\mathsf{HL}_1(\downarrow)$ with an elementary blow-up.

3 Decision Procedures

In this Section, we describe an automata-theoretic approach to solve satisfiability and (finite-state) model checking of HL based on a *direct* translation of HL formulas into a subclass of generalized Büchi alternating word automata. The proposed translation lead to algorithms for the considered problems which run in space of exponential height equal to the nesting depth of the existential binder operator. Moreover, for formulas containing at least two variables, we show that these algorithms are asymptotically optimal by providing matching lower bounds. Finally, for the fragment HL_1 of HL consisting of formulas with only one variable, we show that the complexity of the considered problems is elementary, and precisely, EXPSPACE-complete. For the upper bound of this last result, we will use the following proposition essentially establishing that nested occurrences of the $\exists$-operator in HL_1 formulas can be avoided at no cost.

Proposition 1. *Given a HL_1 formula φ over AP, one can construct a HL_1^1 formula ψ (without nested occurrences of the $\exists$-operator) over a set of propositions $\widehat{AP} \supseteq AP$ such that $|\psi| = O(|\varphi|)$ and for each infinite word w over 2^{AP}, $(w,0) \models \varphi$ iff there is an infinite word $\widehat{w}$ over $2^{\widehat{AP}}$ such that $(\widehat{w},0) \models \psi$ and for each $i \geq 0$, $\widehat{w}(i) \cap AP = w(i)$.*

Proof. By a trivial readaptation of the construction used in [SW07] to show that satisfiability of $\mathsf{HL}_1(\downarrow)$ can be linearly reduced to satisfiability of its fragment $\mathsf{HL}_1^1(\downarrow)$. $\square$

3.1 Alternating Automata

In this Subsection, we recall the class of alternating (finite-state) automata on infinite words equipped with a generalized Büchi acceptance condition (generalized Büchi AWA), and focus on a subclass of such automata, introduced here for the first time.

For a finite set X, $\mathcal{B}_p(X)$ denotes the set of positive boolean formulas over X built from elements in X using $\vee$ and $\wedge$ (we also allow the formulas `true` and `false`). A subset Y of X *satisfies* $\theta \in \mathcal{B}_p(X)$ iff the truth assignment assigning `true` to the elements in Y and `false` to the elements of $X \setminus Y$ satisfies θ. The set Y *exactly satisfies* θ if Y satisfies θ and every proper subset of Y does not satisfy θ.

A generalized Büchi AWA is a tuple $\mathcal{A} = \langle \Sigma, Q, Q_0, \delta, \mathcal{F} \rangle$, where Σ is an input alphabet, Q is a finite set of states, $Q_0 \subseteq Q$ is a set of initial states, $\delta : Q \times \Sigma \to \mathcal{B}_p(Q)$ is a transition function, and $\mathcal{F} = \{F_1, \ldots, F_k\}$ is a set of sets of accepting states. For a state q, a q-run of $\mathcal{A}$ over an infinite word $w \in \Sigma^\omega$ is a Q-labeled tree r such that the root is labeled by q and for each node u with label q' (describing a copy of $\mathcal{A}$ in state q' which reads $w(|u|)$, where $|u|$ denotes the distance of node u from the root), there is a (possibly empty) set $H = \{q_1, \ldots, q_n\} \subseteq Q$ exactly satisfying $\delta(q', w(|u|))$ such that u has n children $u_1, \ldots, u_n$, and for each $1 \leq h \leq n$, u_h has label q_h. The run r is *accepting* if for each infinite path $u_0 u_1 \ldots$ in the tree and each accepting component $F \in \mathcal{F}$, there are infinitely many $i \geq 0$ such that the label of u_i is in F. The ω-language of $\mathcal{A}$, $\mathcal{L}(\mathcal{A})$, is the set of $w \in \Sigma^\omega$ such that there is an accepting q_0-run r of $\mathcal{A}$ over w for some $q_0 \in Q_0$.

As we will see in order to capture HL formulas, it suffices to consider a subclass of AWA, we call *AWA with a main path* (MAWA). Formally, a generalized Büchi MAWA is a generalized Büchi AWA $\mathcal{A} = \langle \Sigma, Q, Q_0, \delta, \mathcal{F} \rangle$ satisfying the following additional conditions. The set of states Q is partitioned into a set Q_m of *main states* and in a set Q_s of *secondary* states. Moreover, $Q_0 \subseteq Q_m$, and for all $\sigma \in \Sigma$, $q_s \in Q_s$, and $q_m \in Q_m$: (i) $\delta(q_s, \sigma)$ does not contain occurrences of main states, and (ii) $\delta(q_m, \sigma)$ is in (positive) disjunctive normal form and there is exactly one main state in each disjunct (hence, a set $Y \subseteq Q$ exactly satisfies $\delta(q_m, \sigma)$ *only if* $Y \cap Q_m$ is a singleton). These requirements ensure that in every run of $\mathcal{A}$, there is exactly one path π (the *main* path) which visits only nodes labeled by main states, and each node that is not visited by π is labeled by a secondary state. By a readaptation of the standard construction used to convert a Büchi AWA into an equivalent standard Büchi nondeterministic word automaton (Büchi NWA) with a single exponential-time blow-up [MH84], we obtain the following result (for details, see [BL08]).

Theorem 1. *Given a generalized Büchi* MAWA $\mathcal{A}$ *with set of states* $Q = Q_m \cup Q_s$ *and acceptance condition* $\mathcal{F} = \{F_1, \dots, F_k\}$*, one can construct a Büchi* NWA $\mathcal{A}_N$ *with number of states* $O(k \cdot |Q_m| \cdot 2^{O(k|Q_s|)})$ *such that* $\mathcal{L}(\mathcal{A}_N) = \mathcal{L}(\mathcal{A})$*.*

3.2 Upper Bounds

In this subsection we describe an automata-theoretic algorithm to solve satisfiability and (finite-state) model-checking of HL. First, we give a non-trivial characterization of the satisfaction relation $(w, 0) \models \varphi$, for a given formula φ, in terms of sequences of sets associated with φ (which generalize the classical notion of Hintikka-set of LTL) satisfying determined requirements which can be checked by generalized Büchi MAWA. Then, we describe the translation into MAWA based on this characterization.

Fix $n \geq 1$ and an alphabet $\Sigma = 2^{AP}$, and let $[n] = \{1, \dots, n\}$. In the following, we consider (possibly open) formulas φ in HL_n over AP. A formula ψ is said to be a *first-level subformula of* φ if there is an occurrence of ψ in φ which is *not* in the scope of the $\exists$-operator. The *closure* $\mathsf{cl}(\varphi)$ of φ is the smallest set containing $\top$, the propositions in AP, the variable x_h for each $h \in [n]$, $\mathsf{X}^-\top$, all the *first-level* subformulas of φ, $\mathsf{X}^{dir}(\psi_1 \,\mathsf{U}^{dir} \psi_2)$ for any *first-level* subformula $\psi_1 \,\mathsf{U}^{dir} \psi_2$ of φ with $dir \in \{+, -\}$, and the negations of all these formulas (we identify $\neg\neg\psi$ with ψ). Note that $|\mathsf{cl}(\varphi)| = O(|\varphi|)$.

Now, we define by induction on $\mathsf{d}_\exists(\varphi)$ the set $\mathsf{Atoms}(\varphi)$ of *atoms* of φ: $A \in \mathsf{Atoms}(\varphi)$ iff $A \subseteq \mathsf{cl}(\varphi) \cup \bigcup_{h \in [n]} (\{x_h\} \times \{-, 0, +\}) \cup \bigcup_{\exists x_h.\psi \in \mathsf{cl}(\varphi)} (\mathsf{Atoms}(\psi) \times \{\psi\} \times \{h\})$, and the following holds (where $dir \in \{+, -\}$):

1. $\top \in A$;
2. for each $\psi \in \mathsf{cl}(\varphi)$, $\psi \in A$ iff $\neg\psi \notin A$;
3. for each $\psi_1 \wedge \psi_2 \in \mathsf{cl}(\varphi)$, $\psi_1 \wedge \psi_2 \in A$ iff $\psi_1, \psi_2 \in A$;
4. for each $\psi_1 \,\mathsf{U}^{dir} \psi_2 \in \mathsf{cl}(\varphi)$, $\psi_1 \,\mathsf{U}^{dir} \psi_2 \in A$ iff or $\psi_2 \in A$ or $\psi_1, \mathsf{X}^{dir}(\psi_1 \,\mathsf{U}^{dir} \psi_2) \in A$;
5. if $\mathsf{X}^-\psi \in A$, then $\mathsf{X}^-\top \in A$;
6. for each $h \in [n]$, $x_h \in A$ iff $(x_h, 0) \in A$;
7. for each $h \in [n]$, A contains exactly one pair of the form $(x_h, dir) \in A$ for some $dir \in \{+, -, 0\}$;
8. if $\mathsf{X}^-\top \notin A$, then for each $h \in [n]$, $(x_h, -) \notin A$;
9. if $(B, \psi, h) \in A$, then $B \cap AP = A \cap AP$, ($\mathsf{X}^-\top \in B$ iff $\mathsf{X}^-\top \in A$), and for each $k \in [n]$ with $k \neq h$ and $dir \in \{+, -, 0\}$, $(x_k, dir) \in B$ iff $(x_k, dir) \in A$;
10. for each $\exists x_h.\psi \in \mathsf{cl}(\varphi)$, there is $(B, \psi, h) \in A$ such that $x_h \in B$;
11. for each $\exists x_h.\psi \in \mathsf{cl}(\varphi)$, $\exists x_h.\psi \in A$ iff there is $(B, \psi, h) \in A$ with $\psi \in B$.

Intuitively, an atom of φ describes a maximal set of subformulas of φ which can hold at a position i of a word $w \in (2^{AP})^\omega$ w.r.t. a determined valuation $j_1, \dots, j_n$ of variables $x_1, \dots, x_n$. In particular, for each $h \in [n]$, the unique pair $(x_h, dir) \in A$ keeps track whether the position j_h referenced by x_h strictly precedes $(dir = -)$, strictly follows $(dir = +)$, or coincides ($dir = 0$ and $x_h \in A$) with the current position i. Finally, a triple

$(B,\psi,h) \in A$, where $\exists x_h.\psi \in \mathsf{cl}(\varphi)$, describes the set of subformulas of ψ which hold at position i w.r.t. a valuation of variables $x_1,\ldots,x_n$ of the form $j_1,\ldots,j_{h-1},m,j_{h+1},\ldots,j_n$ for some $m \in \mathbb{N}$. Thus, $\exists x_h.\psi$ holds at position i w.r.t. the valuation $j_1,\ldots,j_n$ iff $\psi \in B$ for some $(B,\psi,h) \in A$ (Property 11). Note that Property 10 ensures that there is a triple $(B,\psi,h) \in A$ describing the set of subformulas of ψ which hold at position i w.r.t. the valuation of variables $x_1,\ldots,x_n$ given by $j_1,\ldots,j_{h-1},i,j_{h+1},\ldots,j_n$. The necessity of Property 10 will be clear in the proof of Theorem 2 below. Assuming w.l.o.g. that each $p \in AP$ occurs in φ, and $x_1,\ldots,x_n$ occur in φ, by construction it easily follows that $|\mathsf{Atoms}(\varphi)| = \mathsf{Tower}(O(|\varphi|),\mathsf{d}_\exists(\varphi)+1)$.

Now, we define by induction on $\mathsf{d}_\exists(\varphi)$ the function Succ_φ which maps each atom $A \in \mathsf{Atoms}(\varphi)$ to a subset of $\mathsf{Atoms}(\varphi)$. Intuitively, if A is the atom associated with the current position i of the given word w, then $\mathsf{Succ}_\varphi(A)$ contains the set of atoms associable to the next position $i+1$ (w.r.t. a given valuation of variables $x_1,\ldots,x_n$). Formally, $A' \in \mathsf{Succ}_\varphi(A)$ iff the following holds

(a) for each $\mathsf{X}^+\psi \in \mathsf{cl}(\varphi)$, $\mathsf{X}^+\psi \in A \Leftrightarrow \psi \in A'$;
(b) for each $\mathsf{X}^-\psi \in \mathsf{cl}(\varphi)$, $\mathsf{X}^-\psi \in A' \Leftrightarrow \psi \in A$;
(c) for each $h \in [n]$, $(x_h,-) \in A'$ iff $(x_h,dir) \in A$ for some $dir \in \{0,-\}$;
(d) for each $h \in [n]$, $(x_h,+) \in A$ iff $(x_h,dir) \in A'$ for some $dir \in \{0,+\}$;
(e) for each $(B,\psi,h) \in A$, there is $(B',\psi,h) \in A'$ such that $B' \in \mathsf{Succ}_\psi(B)$;
(f) for each $(B',\psi,h) \in A'$, there is $(B,\psi,h) \in A$ such that $B' \in \mathsf{Succ}_\psi(B)$.

For $A \in \mathsf{Atoms}(\varphi)$, let $\sigma(A) = A \cap AP$, i.e. the set of propositions in AP occurring in A. For an infinite word w over $\Sigma = 2^{AP}$, $i \in \mathbb{N}$, and $\widehat{j} \in \mathbb{N} \cup \{\infty\}$ such that $i \leq \widehat{j}$, a $(i,\widehat{j},\varphi)$-*path over* w is a sequence of atoms of φ, $\pi = A_i, A_{i+1},\ldots$ of length $\widehat{j}-i+1$ satisfying the following: for each $i \leq l \leq \widehat{j}$, $\sigma(A_l) = w(l)$ and if $l < \widehat{j}$, then $A_{l+1} \in \mathsf{Succ}_\varphi(A_l)$.

If $i=0$ and $\widehat{j}=\infty$, then π is simply called φ-path over w. A φ-path $\pi = A_0,A_1,\ldots$ over $w \in \Sigma^\omega$ is *fair* iff the following is inductively satisfied:

1. For each $\psi_1 \mathsf{U}^+ \psi_2 \in \mathsf{cl}(\varphi)$, there are infinitely many $i \geq 0$ such that either $\psi_2 \in A_i$ or $\neg(\psi_1 \mathsf{U}^+ \psi_2) \in A_i$;
2. There is $K \geq 0$ such that for each $h \in [n]$, $(x_h,-) \in A_K$, and for all $i \geq K$ and $(B,\psi,h) \in A_i$, there is a *fair* ψ-path starting from B over the suffix w^i of w.

Note that the definition of *fair* φ-path ensures the following important requirement whose proof is immediate.

Lemma 1. *If* $\pi = A_0,A_1,\ldots,A_i$ *is a* $(0,i,\varphi)$*-path on* $w \in \Sigma^\omega$ *and* $\pi' = A_i,A_{i+1},\ldots$ *is a fair* φ*-path on* w^i*, then* $\pi \cdot \pi' = A_0,\ldots,A_i,A_{i+1},\ldots$ *is a* fair φ*-path on* w*.*

Let $\pi = A_0,A_1,\ldots$ be a φ-path over an infinite word w. By Properties 6 and 7 in def. of atom and Properties (c) and (d) in def. of Succ_φ, it holds that for each $h \in [n]$, the set $P_h = \{j \in \mathbb{N} \mid x_h \in A_j\}$ is either empty or a singleton. We say that π is *good* if for each $h \in [n]$, the set P_h is a singleton. Note that by Property 8 in def. of atom, Property (c) in def. of Succ_φ, and Property 2 in def. of *fair* φ-path, the following holds:

Lemma 2. *If* $\pi = A_0,A_1,\ldots,A_i$ *is a fair* φ*-path on* w *with* $\neg\mathsf{X}^-\top \in A_0$*, then* π *is good.*

Now, we prove the main results of this Subsection. First, we need the following lemma.

Lemma 3. *Let $\pi = A_0, A_1, \ldots$ be a fair φ-path over $w \in \Sigma^\omega$ with $\neg X^- \top \in A_0$. Then, for all $i \geq 0$, $m \geq i$ and $(B, \psi, h) \in A_i$, there is a fair ψ-path $\nu = B_0, B_1, \ldots$ over w such that $\neg X^- \top \in B_0$, $B_i = B$, $(B_j, \psi, h) \in A_j$ for all $j \leq m$, and for each $k \in [n] \setminus \{h\}$ and $l \geq 0$, $x_k \in B_l$ iff $x_k \in A_l$.*

Proof. Fix $i \geq 0$, $m \geq i$, and $(B, \psi, h) \in A_i$. Let $K \geq 0$ be the constant (depending on π) of Property 2 in def. of fair φ-path, and let $H \geq max\{m, K\}$. Since $(B, \psi, h) \in A_i$, by Properties (e) and (f) in def. of Succ_φ and Properties 6, 7, and 9 in def. of atom, it follows that there is a $(0, H, \psi)$-path $\rho = B_0, B_1, \ldots, B_H$ over w such that $B_i = B$, $\neg X^- \top \in B_0$, and for each $0 \leq j \leq H$, $(B_j, \psi, h) \in A_j$ and ($x_k \in B_j$ iff $x_k \in A_j$) for all $k \in [n] \setminus \{h\}$. Since $(B_H, \psi, h) \in A_H$ and $H \geq K$, by Property 2 in def. of fair φ-path, there is a fair ψ-path of the form $\rho' = B_H, B_{H+1}, \ldots$ over the suffix w^H of w. Moreover, for each $k \in [n]$, $(x_k, -) \in A_K$. By Property (c) in def. of Succ_φ and Properties 6, 7, and 9 in def. of atom, we obtain that $x_k \notin B_j$ and $x_k \notin A_j$ for all $j \geq H$ and $k \in [n] \setminus \{h\}$. Thus, by Lemma 1 (which holds for any formula ψ in HL_n) it follows that $\rho \cdot \rho' = B_0, \ldots, B_H, B_{H+1}, \ldots$ is a fair ψ-path over w satisfying the statement of the lemma. □

Theorem 2 (Correctness). *Let $\pi = A_0, A_1, \ldots$ be a fair φ-path on $w \in \Sigma^\omega$ such that $\neg X^- \top \in A_0$, and for each $h \in [n]$, let j_h be the unique index such that $x_h \in A_{j_h}$. Then, for each $i \geq 0$ and $\psi \in \mathsf{cl}(\varphi)$, $(w, i, j_1, \ldots, j_n) \models \psi \Leftrightarrow \psi \in A_i$.*

Proof. By induction on $\mathsf{d}_\exists(\varphi)$. The base step ($\mathsf{d}_\exists(\varphi) = 0$) and the induction step ($\mathsf{d}_\exists(\varphi) > 0$) are similar, and we focus on the induction step. Thus, we can assume that the theorem holds for each formula θ such that $\exists x_h.\theta \in \mathsf{cl}(\varphi)$ for some $h \in [n]$ (note that if $\mathsf{d}_\exists(\varphi) = 0$, there is no such formula). Fix $i \geq 0$ and $\psi \in \mathsf{cl}(\varphi)$. By a nested induction on the structure of ψ, we show that $(w, i, j_1, \ldots, j_n) \models \psi \Leftrightarrow \psi \in A_i$. The cases where ψ is a proposition in AP, or ψ has a PLTL operator at its root are managed in a standard way (for details, see [BL08]). For the remaining cases, we proceed as follows:

Case $\psi = x_h$ with $h \in [n]$. $(w, i, j_1, \ldots, j_n) \models x_h \Leftrightarrow i = j_h \Leftrightarrow$ (by def. of j_h) $x_h \in A_i$.

Case $\psi = \exists x_h.\psi_1$ with $h \in [n]$. First, we show the direct implication $(w, i, j_1, \ldots, j_n) \models \psi \Rightarrow \psi \in A_i$. Let $(w, i, j_1, \ldots, j_n) \models \psi$. Then, $(w, i, j_1, \ldots, j_{h-1}, l, j_{h+1}, \ldots, j_n) \models \psi_1$ for some $l \in \mathbb{N}$. By Property 10 in def. of atom there is $(B, \psi_1, h) \in A_l$ such that $x_h \in B$. Let $m \geq \{i, l\}$. Since $\neg X^- \top \in A_0$, by Lemma 3 there is a fair ψ_1-path $\rho = B_0, B_1, \ldots$ over w such that $B_l = B$ (hence, $x_h \in B_l$), $\neg X^- \top \in B_0$, $(B_j, \psi, h) \in A_j$ for each $j \leq m$ (hence, $(B_i, \psi, h) \in A_i$), and for each $k \in [n] \setminus \{h\}$, $x_k \in B_{j_k}$. Since the theorem holds for ψ_1 and $(w, i, j_1, \ldots, j_{h-1}, l, j_{h+1}, \ldots, j_n) \models \psi_1$, it follows that $\psi_1 \in B_i$. Since $(B_i, \psi_1, h) \in A_i$, by Property 11 in def. of atom we obtain that $\psi \in A_i$.

For the converse implication, let $\psi \in A_i$. By Property 11 in def. of atom there is $(B, \psi_1, h) \in A_i$ with $\psi_1 \in B$. Since $\neg X^- \top \in A_0$, by Lemma 3 there is a fair ψ_1-path $\rho = B_0, B_1, \ldots$ over w such that $B_i = B$, $\neg X^- \top \in B_0$, and for each $k \in [n] \setminus \{h\}$, $x_k \in B_{j_k}$. Let $l \in \mathbb{N}$ be the unique index such that $x_h \in B_l$. Since the theorem holds for ψ_1 and $\psi_1 \in B_i$, we obtain that $(w, i, j_1, \ldots, j_{h-1}, l, j_{h+1}, \ldots, j_n) \models \psi_1$, hence $(w, i, j_1, \ldots, j_n) \models \psi$. □

Moreover, we can show the following result (a proof is in [BL08]).

Theorem 3 (Completeness). *Let $w \in \Sigma^\omega$ and $j_1, \ldots, j_n \in \mathbb{N}$. Then, there is a fair φ-path $\pi = A_0, A_1, \ldots$ over w such that $\neg X^- \top \in A_0$ and for each $k \in [n]$, $x_k \in A_{j_k}$.*

By Theorems 2 and 3 we obtain the following characterization of $(w,0) \models \varphi$.

Corollary 1. *For each word* $w \in \Sigma^\omega$, $(w,0) \models \varphi$ *iff there is a fair* φ*-path* $\pi = A_0, A_1, \ldots$ *over* w *such that* $\varphi, \neg\mathsf{X}^-\top, x_h \in A_0$ *for each* $h \in [n]$.

Translation into MAWA Now, we illustrate the translation of HL formulas into generalized Büchi MAWA based on the result of Corollary 1.

Theorem 4. *For a HL formula* φ *over AP, one can build a generalized Büchi MAWA* $\mathcal{A}_\varphi$ *over* 2^{AP} *with states* $Q_m \cup Q_s$ *and* $O(|\varphi|)$ *Büchi components such that* $\mathcal{L}(\mathcal{A}_\varphi) = \mathcal{L}(\varphi)$, $|Q_m| = \mathsf{Tower}(O(|\varphi|), \mathsf{d}_\exists(\varphi)+1)$, *and* $|Q_s| = \mathsf{Tower}(O(|\varphi|), \mathsf{d}_\exists(\varphi))$.

Proof. Let $x_1, \ldots, x_n$ be the variables occurring in φ. We construct a generalized Büchi MAWA $\mathcal{A}_\varphi$ of the desired size with set of main states containing $\mathsf{Atoms}(\varphi)$ and set of initial states given by $\{A \in \mathsf{Atoms}(\varphi) \mid \varphi, \neg\mathsf{X}^-\top, x_h \in A$ for each $h \in [n]\}$ such that for each $A \in \mathsf{Atoms}(\varphi)$ and $w \in \Sigma^\omega$, $\mathcal{A}_\varphi$ has an accepting A-run over w iff there is a fair φ-path over w starting from A. Hence, the result follows from Corollary 1. The construction is given by induction on $\mathsf{d}_\exists(\varphi)$. Thus, we can assume that for each $\exists x_h.\psi \in \mathsf{cl}(\varphi)$, one can construct the MAWA $\mathcal{A}_\psi$ associated with ψ. Here, we informally describe the construction (the formal definition is in [BL08]).

Assume that $\mathcal{A}_\varphi$ starts the computation over an input w in a main state $A \in \mathsf{Atoms}(\varphi)$. Then, $\mathcal{A}_\varphi$ guesses a φ-path $\pi = A_0, A_1, \ldots$ over w (with $A_0 = A$) by simulating it along the main path of the run tree. In order to check that π satisfies Property 2 in def. of fair φ-path, $\mathcal{A}_\varphi$ guesses a point along the main path (the constant K in Property 2), checks that $(x_h, -)$ is in the current guessed atom for each $h \in [n]$, and from this instant forward, for each triple (B, ψ, h) in the current guessed atom, $\mathcal{A}_\varphi$ starts an additional copy of the MAWA $\mathcal{A}_\psi$ in state B (which represents a secondary $\mathcal{A}_\varphi$-state). Finally, the acceptance condition of $\mathcal{A}_\varphi$ extends the acceptance conditions of the MAWAs $\mathcal{A}_\psi$ with additional sets used to check that the infinite sequence of states visited by the main copy of $\mathcal{A}_\varphi$ (corresponding to the simulated φ-path) satisfies Property 1 in def. of fair φ-path. □

Corollary 2. *Given a HL formula* φ *over AP, one can build a Büchi NWA* $\mathcal{A}_{N,\varphi}$ *over* 2^{AP} *of size* $\mathsf{Tower}(O(|\varphi|), \mathsf{d}_\exists(\varphi)+1)$ *s.t.* $\mathcal{L}(\mathcal{A}_{N,\varphi}) = \mathcal{L}(\varphi)$. *Moreover, if* φ *is an* HL_1 *formula, then one can build an equivalent Büchi NWA of size doubly exponential in* $|\varphi|$.

Proof. The first result follows from Theorems 1 and 4. For the second one, note that for $\widehat{AP} \supset AP$ and a Büchi NWA $\mathcal{A}_N$ over $2^{\widehat{AP}}$, $\mathcal{L}(\mathcal{A}_N)$ can be seen as a language on $2^{AP} \times 2^{\widehat{AP} \setminus AP}$. Since one can build a Büchi NWA of the same size as $\mathcal{A}_N$ accepting the projection of $\mathcal{L}(\mathcal{A}_N)$ on 2^{AP}, the result follows from Proposition 1 and Theorems 1 and 4. □

By Corollary 2, the model checking problem for an HL formula φ is reduced to emptiness of the Büchi NWA $\mathcal{A}_{\neg\varphi,\mathcal{K}}$ obtained as the synchronous product of the Büchi NWA $\mathcal{A}_\mathcal{K}$ (where each state is accepting) corresponding to the given Kripke structure $\mathcal{K}$ and the Büchi NWA $\mathcal{A}_{N,\neg\varphi}$ associated with the *negation* of formula φ. Since nonemptiness of Büchi NWA is in NLogspace, by Corollary 2 we obtain the following result.

Theorem 5 (Upper bounds). *Satisfiability and model checking of* HL_1 *and* HL^h *(for any* $h \geq 1$*) are in* Expspace *and* h-Expspace, *respectively.*

3.3 Lower Bounds

In this Subsection we show that for each $h \geq 1$, satisfiability and model-checking of $\mathsf{HL}_2^h(\downarrow)$ are h-EXPSPACE-hard by a reduction from the word problem for $\exp[h]$-space bounded deterministic Turing Machines. In the following, w.l.o.g. we assume that the considered HL formulas φ over a finite set of propositions AP are interpreted on words over 2^{AP} where each symbol is a singleton (these words can be seen as words over AP).

Fix $n \geq 1$, a finite alphabet $\Sigma \cup \{0,1\}$, and a countable set $\{\$_1, \$_2, \ldots\}$ of symbols non in $\Sigma \cup \{0,1\}$. First, for each $h \geq 1$, we define by induction on h an encoding of the integers in $[0, \mathsf{Tower}(n,h) - 1]$ by finite words, called (h,n)*-codes*, over $\{\$_1, \ldots, \$_h, 0, 1\}$ of the form $\$_h w \$_h$, where w does not contain occurrences of $\$_h$.

Base Step. $h = 1$. A $(1,n)$*-block* over Σ is a finite word w over $\{\$_1, 0, 1\} \cup \Sigma$ having the form $w = \$_1 \sigma b_1 \ldots b_n \$_1$, where $\sigma \in \Sigma \cup \{0,1\}$ and $b_1, \ldots, b_n \in \{0,1\}$. The *block-content* $\mathsf{CON}(w)$ of w is σ, and the *block-number* $\mathsf{NUM}(w)$ of w is the natural number in $[0, \mathsf{Tower}(n,1) - 1]$ (recall that $\mathsf{Tower}(n,1) = 2^n$) whose binary code is $b_1 \ldots b_n$. An $(1,n)$*-code* is a $(1,n)$-block w such that $\mathsf{CON}(w) \in \{0,1\}$.

Induction Step. let $h \geq 1$. A $(h+1,n)$*-block* on Σ is a word w on $\{\$_1, \ldots, \$_{h+1}, 0, 1\} \cup \Sigma$ of the form $\$_{h+1} \sigma \$_h w_1 \$_h w_2 \$_h \ldots \$_h w_K \$_h \$_{h+1}$, where $\sigma \in \{0,1\} \cup \Sigma$, $K = \mathsf{Tower}(n,h)$ and for each $1 \leq i \leq K$, $\$_h w_i \$_h$ is a (h,n)*-code* such that $\mathsf{NUM}(\$_h w_i \$_h) = i - 1$. The *block-content* $\mathsf{CON}(w)$ of w is the symbol σ, and the *block-number* $\mathsf{NUM}(w)$ of w is the natural number in $[0, \mathsf{Tower}(n,h+1) - 1]$ whose binary code is given by $\mathsf{CON}(\$_h w_1 \$_h) \ldots \mathsf{CON}(\$_h w_K \$_h)$. A $(h+1,n)$*-code* is a $(h+1,n)$-block w such that $\mathsf{CON}(w) \in \{0,1\}$.

For each $h \geq 1$, a (h,n)-configuration over Σ is a finite word w of the form $w = \$_{h+1} \$_h w_1 \$_h w_2 \$_h \ldots \$_h w_K \$_h \$_{h+1}$, where $K = \mathsf{Tower}(n,h)$ and for any $1 \leq i \leq K$, $\$_h w_i \$_h$ is a (h,n)*-block* such that $\mathsf{NUM}(\$_h w_i \$_h) = i - 1$ and $\mathsf{CON}(\$_h w_i \$_h) \in \Sigma$. As we will see, (h,n)-configurations are used to encode the configurations reachable by $\exp[h]$-space bounded deterministic Turing machines on inputs of size n.

We will use the following non-trivial technical result, whose proof is in [BL08], where for each $h \geq 1$, $\mathsf{Parity}(h) := 1$ if h is odd, and $\mathsf{Parity}(h) := 2$ otherwise.

Proposition 2. *For each $h \geq 1$, we can construct two $\mathsf{HL}_2^{h-1}(\downarrow)$ formulas ψ_h^{conf} and $\psi_h^{=}$ over $\{\$_1, \ldots, \$_{h+1}, 0, 1\} \cup \Sigma$ of sizes bounded by $O(n^3 \cdot h \cdot |\Sigma|)$ such that $\psi_h^{=}$ is open and for $w \in \Sigma^\omega$ and $i \geq 0$, we have*

- *for all j_1, j_2, $(w, i, j_1, j_2) \models \psi_h^{conf}$ iff w^i has a prefix that is a (h,n)-configuration;*
- *if there is $j > i$ such that $w[i,j] = \$_h w_1 \$_h w' \$_h w_2 \$_h$, where $\$_h w_1 \$_h$ and $\$_h w_2 \$_h$ are (h,n)-blocks over Σ, then for each $m \geq 0$,*
  - ***Case*** $\mathsf{Parity}(h) = 1$*:* $(w, i, j, m) \models \psi_h^{=}$ iff $\mathsf{NUM}(\$_h w_1 \$_h) = \mathsf{NUM}(\$_h w_2 \$_h)$*;*
  - ***Case*** $\mathsf{Parity}(h) = 2$*:* $(w, j, m, i) \models \psi_h^{=}$ iff $\mathsf{NUM}(\$_h w_1 \$_h) = \mathsf{NUM}(\$_h w_2 \$_h)$.

Theorem 6. *For each $h \geq 1$, the satisfiability and model checking problems for $\mathsf{HL}_2^h(\downarrow)$ are both h-EXPSPACE-hard.*

Proof. Fix $h \geq 1$. First, we consider the satisfiability problem for $\mathsf{HL}_2^h(\downarrow)$. Let $\mathcal{M} = \langle A, Q, q_0, \delta, F \rangle$ be an $\exp[h]$-space bounded Turing Machine (TM, for short) without halting configurations, and let $c \geq 1$ be a constant such that for each $\alpha \in A^*$, the space needed by $\mathcal{M}$ on input α is bounded by $\mathsf{Tower}(|\alpha|^c, h)$. For $\alpha \in A^*$, we construct a $\mathsf{HL}_2^h(\downarrow)$ formula $\varphi_{\mathcal{M},\alpha}$ of size *polynomial* in $n = |\alpha|^c$ and in the size of $\mathcal{M}$, such that $\mathcal{M}$ accepts α *iff* $\varphi_{\mathcal{M},\alpha}$ is satisfiable.

Note that any reachable configuration of $\mathcal{M}$ over α can be seen as a word $\alpha_1 \cdot (q,a) \cdot \alpha_2$ in $A^* \cdot (Q \times A) \cdot A^*$ of length $\mathsf{Tower}(n,h)$, where $\alpha_1 \cdot a \cdot \alpha_2$ denotes the tape content, q the current state, and the reading head is at position $|\alpha_1|+1$. If $\alpha = a_1 \ldots a_r$ (where $r = |\alpha|$), then the initial configuration is given by $(q_0,a_1)a_2 \ldots a_r \underbrace{\#\#\ldots\#}_{\mathsf{Tower}(n,h)-r}$, where # is the blank symbol. Let $C = u_1 \ldots u_{\mathsf{Tower}(n,h)}$ be a TM configuration. For $1 \leq i \leq \mathsf{Tower}(n,h)$, the value u'_i of the i^{th} cell of the $\mathcal{M}$-successor of C is completely determined by the values u_{i-1}, u_i and u_{i+1} (taking u_{i+1} for $i = \mathsf{Tower}(n,h)$ and u_{i-1} for $i = 1$ to be some special symbol). Let $next_{\mathcal{M}}(u_{i-1},u_i,u_{i+1})$ be our expectation for u'_i (this function can be trivially obtained from the transition function δ of $\mathcal{M}$).

Let $\Sigma = A \cup (Q \times A)$. The *code* of a TM configuration $C = u_1 \ldots u_{\mathsf{Tower}(n,h)}$ is the (h,n)-configuration over Σ given by $\$_{h+1}\$_h w_1 \$_h \ldots \$_h w_{\mathsf{Tower}(n,h)} \$_h \$_{h+1}$, where for each $1 \leq i \leq \mathsf{Tower}(n,h)$, $\mathsf{CON}(\$_h w_i \$_h) = u_i$. A non-empty finite sequence of TM configurations $\wp = C_0 C_1 \ldots C_m$ is encoded by the infinite word over $\Sigma \cup \{0,1,acc,\$_1,\ldots \$_{h+1}\}$, called *sequence-code*, given by $w_{\wp} = \$_{h+1} w_{C_0} \$_{h+1} \ldots \$_{h+1} w_{C_m} \$_{h+1} (acc)^{\omega}$, where for each $0 \leq i \leq m$, $\$_{h+1} w_{C_i} \$_{h+1}$ is the code of configuration C_i. The sequence-code $w_{\wp}$ is *good* iff C_0 is the initial TM configuration over α, C_m is an accepting TM configuration (i.e., the associated state is in F), and $\wp$ is faithful to the evolution of $\mathcal{M}$. Thus, $\mathcal{M}$ accepts α iff there is a good sequence-code.

Now, we build a $\mathsf{HL}_2(\downarrow)$ formula $\varphi_{\mathcal{M},\alpha}$ which is (initially) satisfied by a word w iff w is a good sequence-code. Hence, $\mathcal{M}$ accepts α iff $\varphi_{\mathcal{M},\alpha}$ is satisfiable. Formula $\varphi_{\mathcal{M},\alpha}$ uses the formulas $\psi_h^=$ and ψ_h^{conf} of Proposition 2 (for fixed n and Σ), and is given by

$$\varphi_{\mathcal{M},\alpha} = \varphi_{SC} \wedge \varphi_{first} \wedge \varphi_{acc} \wedge \varphi_{\delta}$$

where: (1) φ_{SC} uses ψ_h^{conf} and checks that the given word is a sequence-code of some sequence of TM configurations $\wp = C_0, \ldots, C_m$, (2) φ_{first} is a PLTL formula checking that C_0 is the initial configuration, (3) φ_{acc} is a PLTL formula checking that C_m is an accepting configuration, and (4) φ_{δ} uses $\psi_h^=$ and checks that $\wp$ is faithful to the evolution of $\mathcal{M}$. The construction of φ_{first} and φ_{acc} is simple. Thus, we focus on φ_{SC} and φ_{δ}.

$$\varphi_{SC} = \$_{h+1} \wedge (\mathsf{X}^+ \neg acc) \wedge ((\$_{h+1} \rightarrow \underline{\psi_h^{conf}})\, \mathsf{U}^+ (\$_{h+1} \wedge \mathsf{G}^+ \mathsf{X}^+ acc))$$

(recall that $(w,i,j_1,j_2) \models \psi_h^{conf}$ iff w^i has a prefix that is a (h,n)-configuration over Σ).

Finally, we define formula φ_{δ}, which uses $\psi_h^=$. Here, we assume that $\mathsf{Parity}(h) = 1$ (the other case being similar). Recall that if $\mathsf{Parity}(h) = 1$, then for each subword $w[i,j]$ of the given word w and $m \geq 0$ such that $w[i,j] = bl \cdot w' \cdot bl'$, where bl and bl' are (h,n)-blocks, then $(w,i,j,m) \models \psi_h^=$ iff $\mathsf{NUM}(bl) = \mathsf{NUM}(bl')$.

For a sequence-code w, we have to require that for each subword $\$_{h+1} w_1 \$_{h+1} w_2 \$_{h+1}$, where $\$_{h+1} w_1 \$_{h+1}$ and $\$_{h+1} w_2 \$_{h+1}$ encode two TM configurations C_1 and C_2, C_2 is the TM successor of C_1, i.e., for each (h,n)-block bl' of $\$_{h+1} w_2 \$_{h+1}$, the block-content u' of bl' satisfies $u' = next_{\mathcal{M}}(u_p,u,u_s)$, where u is the block-content of the (h,n)-block bl of $\$_{h+1} w_1 \$_{h+1}$ *having the same block-number as* bl', and u_p (resp., u_s) is the block-content of the (h,n)-block of $\$_{h+1} w_1 \$_{h+1}$ — if any — that precedes (resp., follows)

bl. We define only the formula which encodes the case in which bl' is a non-extremal (h,n)-block. The other cases can be handled similarly. Such a formula is defined as follows, where for $u' \in \Sigma$, $H(u')$ is the set of triples $(u_p,u,u_s) \in [\Sigma]^3$ such that $u' = next_{\mathcal{M}}(u_p,u,u_s)$:

$$\bigwedge_{u' \in \Sigma} \mathsf{G}^+\Big(\{u' \wedge (\neg \$_h \mathsf{U}^+(\$_h \wedge \mathsf{X}^+ \neg \$_{h+1})) \wedge (\neg \mathsf{X}^{-3} \$_{h+1}) \wedge \mathsf{F}^-(\$_{h+1} \wedge \mathsf{X}^- \top)\} \longrightarrow \{\neg \$_h \, \mathsf{U}^+ (\$_h \wedge \bigvee_{(u_p,u,u_s) \in H(u')} \downarrow x_1 . \psi_\delta^{u_p,u,u_s})\}\Big)$$

$$\psi_\delta^{u_p,u,u_s} := \mathsf{F}^- \Big(u_p \wedge \{\neg \$_h \, \mathsf{U}^+(\$_h \wedge \underline{\psi_h^=} \wedge \mathsf{X}^+(u \wedge (\neg \$_h \, \mathsf{U}^+ (\$_h \wedge \mathsf{X}^+ u_s))))\} \wedge \{\neg \$_{h+1} \, \mathsf{U}^+ (\$_{h+1} \wedge \mathsf{X}^+(\neg \$_{h+1} \, \mathsf{U}^+ x_1))\}\Big)$$

By Proposition 2 it follows that $\varphi_{\mathcal{M},\alpha}$ is a $\mathsf{HL}_2^h(\downarrow)$ formula of size polynomial in the size of α and $\mathcal{M}$. Model-checking for $\mathsf{HL}_2^h(\downarrow)$ is also h-EXPSPACE-hard since (1) non-satisfiability is linearly reducible to validity (note that $\mathsf{HL}_2^h(\downarrow)$ is closed under negation), and (2) validity is linearly reducible to model-checking [DS02]. □

Since satisfiability and model checking of $\mathsf{HL}_1(\downarrow)$ are EXPSPACE-hard [SW07], by Theorems 5 and 6, we obtain the following corollary.

Corollary 3. *Satisfiability and model checking of HL_1 are* EXPSPACE*-complete. Moreover, for all $h \geq 1$ and $k \geq 2$, satisfiability and model checking of HL_k^h and $\mathsf{HL}_k^h(\downarrow)$ are h-*EXPSPACE*-complete.*

4 Succinctness Issues

In this Section, we show that for all $h > k \geq 0$, there is a succinctness gap between HL^h and HL^k of exponential height equal to $h-k$. Actually, we show a stronger result: for each $h \geq 1$, there is an alphabet Σ_h and a family of $\mathsf{HL}_2^h(\downarrow)$ formulas $(\varphi_{h,n})_{n \geq 1}$ over Σ_h such that for each $n \geq 1$, $\varphi_{h,n}$ has size polynomial in n and each initially equivalent HL^k formula, for $k < h$, has size at least $\mathsf{Tower}(\Omega(n), h-k)$.

We use the encoding defined in Subsection 3.3. For all $n,h \geq 1$, a *finite (h,n)-good word w* is a finite word over the alphabet $\{0,1,\$_1,\ldots,\$_{h+1}\}$ having the form $w = \$_{h+1} w_1 \$_{h+1} \ldots \$_{h+1} w_m \$_{h+1}$ such that $m > 1$, for each $1 \leq i \leq m$, $\$_{h+1} w_i \$_{h+1}$ is a $(h+1,n)$-code, and the following holds:

Case $\mathsf{Parity}(h) = 1$**.** There is $1 < i \leq m$ s.t. $\mathsf{NUM}(\$_{h+1} w_1 \$_{h+1}) = \mathsf{NUM}(\$_{h+1} w_i \$_{h+1})$;

Case $\mathsf{Parity}(h) = 2$**.** There is $1 \leq i < m$ s.t. $\mathsf{NUM}(\$_{h+1} w_i \$_{h+1}) = \mathsf{NUM}(\$_{h+1} w_m \$_{h+1})$.

An *infinite (h,n)-good word w* is an infinite word of the form $w \cdot \{\#\}^\omega$ such that w is a finite (h,n)-good word. The proofs of the following Lemma 4 (which is based on results of Proposition 2) and Lemma 5 can be found in [BL08].

Lemma 4. *For each $h \geq 1$ and $n \geq 1$, there is a $\mathsf{HL}_2^h(\downarrow)$ formula $\psi_{h,n}^{GOOD}$ of size $O(n^3)$ such that $\mathcal{L}(\psi_{h,n}^{GOOD})$ is the set of infinite (h,n)-good words.*

Lemma 5. *For each $n \geq 1$ and $h \geq 1$, any Büchi* NWA *accepting the set of infinite (h,n)-good words needs at least* $\mathsf{Tower}(n,h+1)$ *states.*

For all $n,h \geq 1$ and $k < h$, let $\psi_{h,n}^{GOOD}$ be the $\mathsf{HL}_2^h(\downarrow)$ formula of Lemma 4, and let φ be an equivalent HL^k formula. By Corollary 2, φ can be translated into an equivalent Büchi NWA $\mathcal{A}_\varphi$ of size $\mathsf{Tower}(O(|\varphi|),k+1)$. By Lemma 5, $\mathcal{A}_\varphi$ has at least $\mathsf{Tower}(n,h+1)$ states, hence $|\varphi|$ is at least $\mathsf{Tower}(\Omega(n),h-k)$. Thus, we obtain the following result.

Theorem 7. *For all $h > k \geq 0$ and $m \geq 2$, there is a succinctness gap between $\mathsf{HL}_m^h(\downarrow)$ and HL^k of exponential height $h-k$.*

5 Conclusions

There are two interesting questions which have been left open in this paper. In the HL_2^h formulas used in the proof of h-EXPSPACE-hardness, there is a strict nested alternation between the binder modalities $\exists x_1$ and $\exists x_2$. Since satisfiability of HL_1 is only EXPSPACE-complete, we conjecture that the considered decision problems can be solved in space of exponential height equal to the depth of nested alternations of binder modalities associated with distinct variables. Another interesting question is to investigate the succinctness gap between HL^k and $\mathsf{HL}^k(\downarrow)$ for each $k \geq 1$.

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