Computational Logic

Constraint Logic Programming

Constraints

- Constraint: conditions that a solution must satisfy
  - $X + Y = 20$
  - $X \wedge Y$ is true
  - The third field of the data structure is greater that the second
  - The murderer is one of those who had met the cabaret entertainer
- CLP: LP plus the ability to compute with some form of constraints (which are solved by the system during computation)
- (Additional) features of a CLP system:
  - Domain of computation (reals, rationals, integers, booleans, structures, $\ldots$)
  - *Expressions* that can be built ($+, *, \wedge, \vee$)
  - *Constraints* allowed: equations, disequations, inequations, etc. ($=, \neq, \leq, \geq, <, >$)
  - *Constraint solving algorithms*: simplex, gauss, etc.
- Solutions: assignments to variables, or new constraints among variables.

A comparison with classic LP (I)

- Example (**plain Prolog**): `q(X, Y, Z):-Z = f(X, Y).`

```
?- q(3, 4, Z).
Z = f(3,4)

?- q(X, Y, f(3,4)).
X = 3, Y = 4

?- q(X, Y, Z).
Z = f(X,Y)
```

- Example (**plain Prolog**): `p(X, Y, Z):-Z is X +Y.`

```
?- p(3, 4, Z).
Z = 7

?- p(X, 4, 7).
{INSTANTIATION ERROR}  ← is/2 not reversible, does not work!
```

A Comparison with classic LP (II)

- Example (**CLP(ℜ) package**):

```
:- use_package(clpr).
p(X, Y, Z) :-  Z .=. X + Y.
```

```
?- p(3, 4, Z).
Z .=. 7

?- p(X, 4, 7).
X .=. 3

4 ?- p(X, Y, 7).
X .=. 7 - Y                          ← with clpr arithmetic is reversible!
```

A Comparison with classic LP (III)

- Advantages:
  - ◇ Helps making programs expressive and flexible.
  - ◇ May save much coding.
  - ◇ In some cases, more efficient than classic LP programs due to solvers typically being very efficiently implemented.
  - ◇ Also, efficiency due to search space reduction:
    - * LP: generate-and-test.
    - * CLP: constrain-and-generate.
- Disadvantages:
  - ◇ Complexity of solver algorithms (simplex, gauss, etc) can affect performance.
- Some solutions:
  - ◇ Better algorithms.
  - ◇ Compile-time optimizations (program transformation, global analysis, etc).
  - ◇ Parallelism.

Example of Search Space Reduction

- Using **plain Prolog** (generate–and–test):

```
% Find three consecutive numbers in the p/1 relation.

solution(X, Y, Z) :-
    p(X), p(Y), p(Z),
    test(X, Y, Z).

p(11). p(3). p(7). p(16). p(15). p(14).

test(X, Y, Z) :-  Y is X + 1, Z is Y + 1.
```

- Query:

```
?- solution(X, Y, Z).
X = 14, Y = 15, Z = 16 ? ;
no
```

- 458 steps (all solutions: 475 steps).

Example of Search Space Reduction

- Using the **CLP($\Re$) package** (generate–and–test):

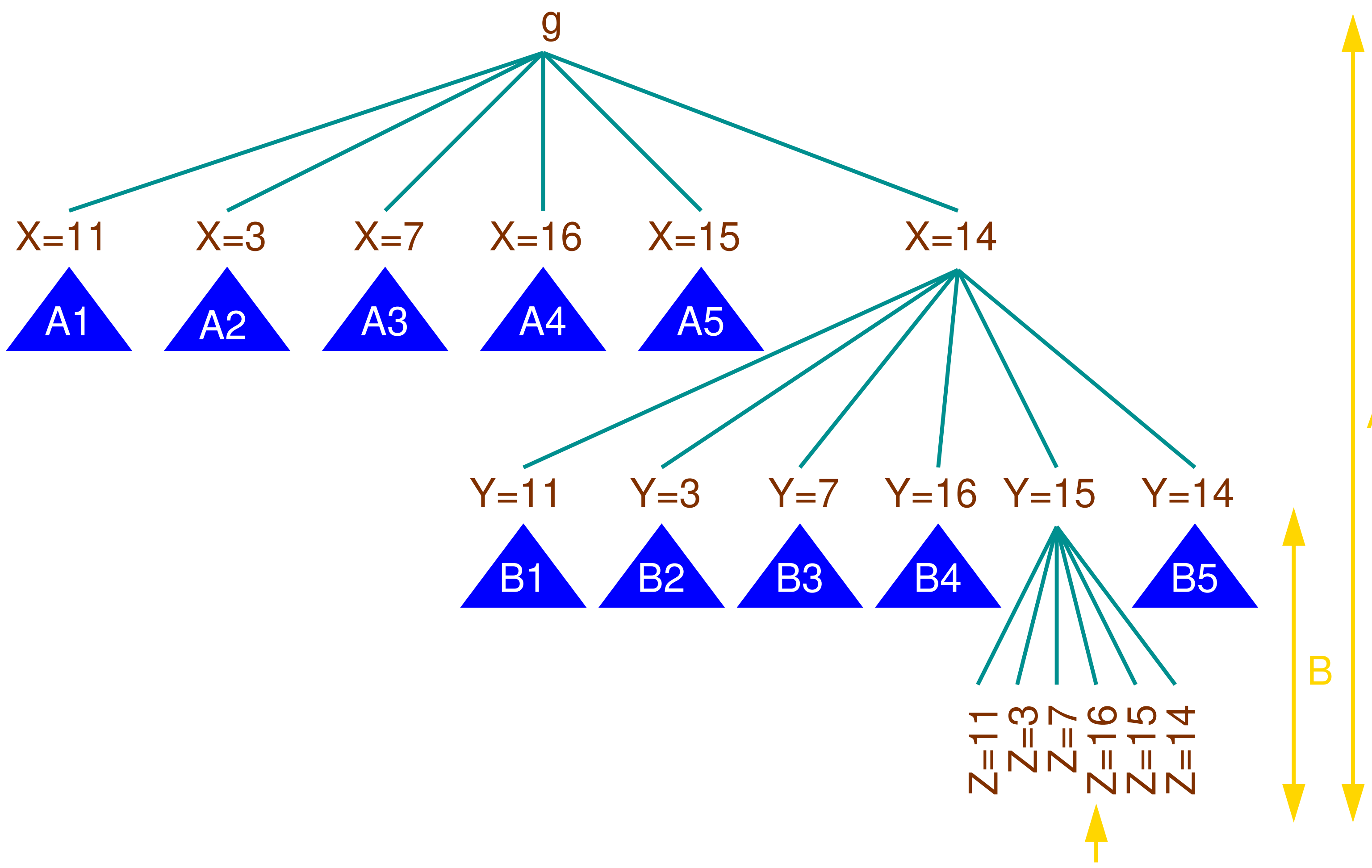
```
% Find three consecutive numbers in the p/1 relation.
:- use_package(clpr).
solution(X, Y, Z) :-
    p(X), p(Y), p(Z),
    test(X, Y, Z).

p(11). p(3). p(7). p(16). p(15). p(14).

test(X, Y, Z) :-  Y .=. X + 1, Z .=. Y + 1.
```

- Query:

```
?- solution(X, Y, Z).
X .=. 14, Y .=. 15, Z .=. 16 ? ;
no
```

- 458 steps (all solutions: 475 steps).

Generate–and–test Search Tree



Example of Search Space Reduction

- *Move* `test(X, Y, Z)` *to the beginning (constrain–and–generate):*

```
% Find three consecutive numbers in the p/1 relation.
:- use_package(clpr).
solution(X, Y, Z) :-
    test(X, Y, Z),
    p(X), p(Y), p(Z).
p(11). p(3). p(7). p(16). p(15). p(14).
```

- Using **plain Prolog:** `test(X, Y, Z):-Y is X +1, Z is Y +1.`

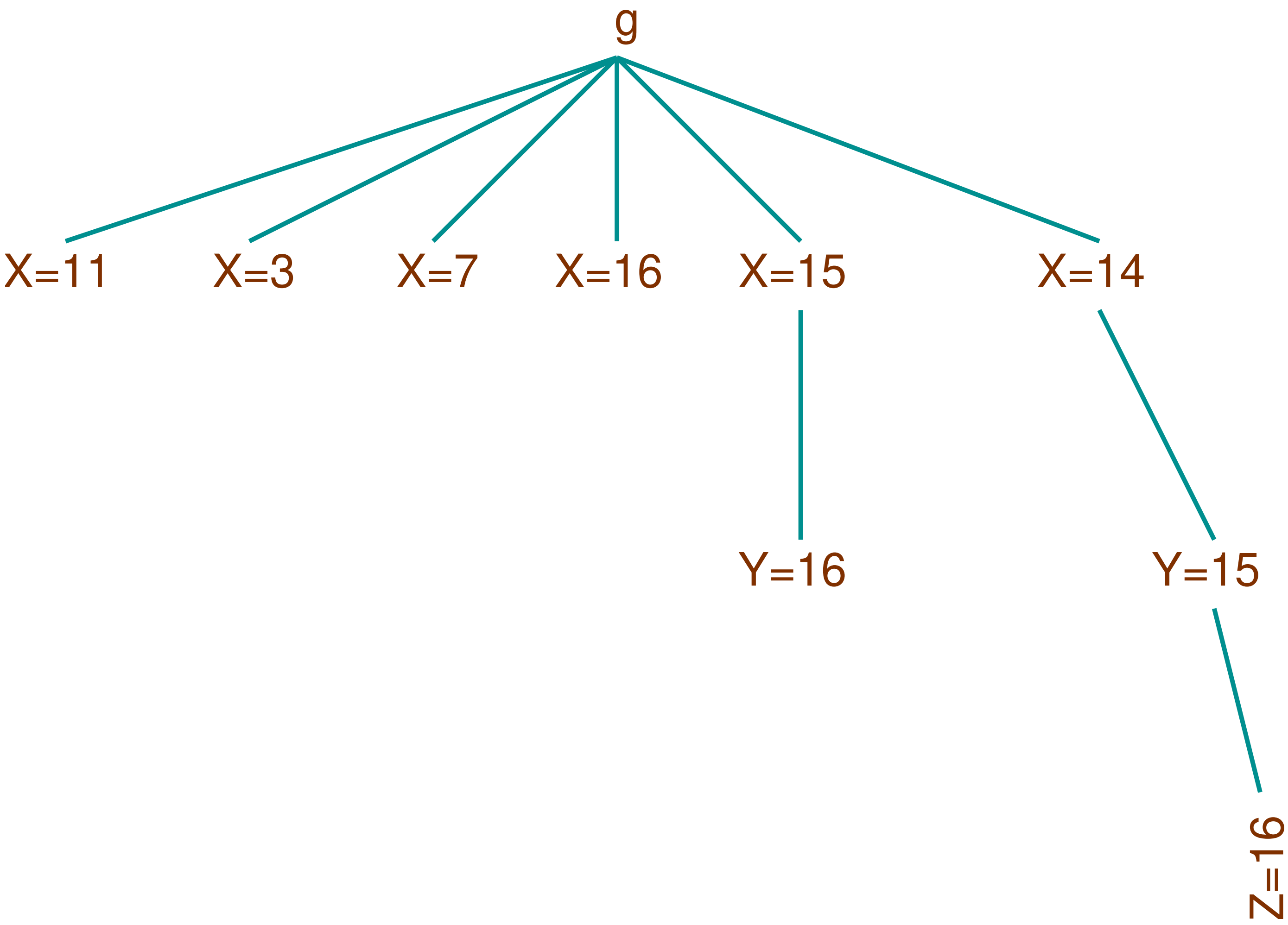
```
?- solution(X, Y, Z).
{INSTANTIATION ERROR}
```

- Using the **CLP($\Re$) package:** `test(X, Y, Z):-Y .=.X +1, Z .=.Y +1.`

```
?- solution(X, Y, Z).
X .=. 14, Y .=. 15, Z .=. 16 ? ;
no
```

In **11 steps** (and all solutions in **11 steps**)!

Constrain–and–generate Search Tree



Constraint Systems: CLP($\mathcal{X}$)

- The semantics is parameterized by the *constraint domain* $\mathcal{X}$:
  CLP($\mathcal{X}$), where $\mathcal{X} \equiv (\Sigma, \mathcal{D}, \mathcal{L}, \mathcal{T})$:
  - ◇ Σ: set of *predicate* and *function symbols*, together with their arity
  - ◇ $\mathcal{L} \subseteq \Sigma$–formulae: constraints
  - ◇ D: the set of actual elements in the constraint domain
  - ◇ $\mathcal{D}$: meaning of predicate and function symbols (and hence, constraints).
  - ◇ $\mathcal{T}$: a first–order theory (axiomatizes some properties of $\mathcal{D}$)
- $(\mathcal{D}, \mathcal{L})$ is a *constraint domain*
- Assumptions:
  - ◇ $\mathcal{L}$ built upon a first–order language
  - ◇ $= \in \Sigma$ and $=$ is *identity* in $\mathcal{D}$
  - ◇ There are identically false and identically true constraints in $\mathcal{L}$
  - ◇ $\mathcal{L}$ is closed w.r.t. renaming, conjunction, and existential quantification

Constraint Domains (I)

- $\Sigma = \{0, 1, +, *, =, <, \leq\}$, D = **R** (the reals), $\mathcal{D}$ interprets Σ as usual, $\Re = (\mathcal{D}, \mathcal{L})$

  → **Arithmetic over the reals** ("$\Re$" domain).
  - ◇ Eg.: $x^2 + 2xy < \frac{y}{x} \wedge x > 0 \;\; (\equiv xxx + xxy + xxy < y \wedge 0 < x)$
  - ◇ Question: is 0 needed? How can it be represented?

- $\Sigma' = \{0, 1, +, =, <, \leq\}$, $\Re_{Lin} = (\mathcal{D}', \mathcal{L}')$

  → **Linear arithmetic** ("$\Re_{Lin}$" domain)
  - ◇ Eg.: $3x - y < 3 \;\; (\equiv x + x + x < 1 + 1 + 1 + y)$

- $\Sigma'' = \{0, 1, +, =\}$, $\Re_{LinEq} = (\mathcal{D}'', \mathcal{L}'')$

  → **Linear equations** ("$\Re_{LinEq}$" domain)
  - ◇ Eg.: $3x + y = 5 \wedge y = 2x$

- A corresponding set of domains can be defined on the **rationals** ("$\mathcal{Q}$" domain)

Constraint Domains (II)

- A very special domain:
  - ◇ $\Sigma = \{<$*constant and function symbols*$>, =\}$
  - ◇ D = { *finite trees* }
  - ◇ $\mathcal{D}$ interprets Σ as tree constructors
    - * Each $f \in \Sigma$ with arity n maps n trees to a tree with root labeled f and whose subtrees are the arguments of the mapping
  - ◇ Constraints: syntactic tree equality
  - ◇ $\mathcal{FT} = (\mathcal{D}, \mathcal{L})$

→ **Equality constraints over the Herbrand domain** ($\mathcal{FT}$ domain)

- ◇ Eg.: $g(h(Z), Y) = g(Y, h(a))$

- LP $\equiv$ CLP($\mathcal{FT}$)
  - ◇ I.e., classical LP can be viewed as constraint logic programming over *Herbrand terms* with a single *constraint predicate symbol*: `=`.

Constraint Domains (III)

- $\Sigma = \{<\textit{constants}>, \lambda, ., ::, =\}$
- D = { finite strings of constants }
- $\mathcal{D}$ interprets . as string concatenation, :: as string length

  → **Equations over strings of constants** ($\mathcal{D}$ domain)

  ◇ Eg.: $X.A.X = X.A$

- $\Sigma = \{0, 1, \neg, \wedge, =\}$
- D = $\{true, false\}$
- $\mathcal{D}$ interprets symbols in Σ as boolean functions
- $\mathcal{BOOL} = (\mathcal{D}, \mathcal{L})$

  → **Boolean constraints** ($\mathcal{BOOL}$ domain)

  ◇ Eg.: $\neg(x \wedge y) = 1$

CLP($\mathcal{X}$) Programs

- Recall that:
  - ◇ Σ is a set of predicate and function symbols
  - ◇ $\mathcal{L} \subseteq \Sigma$–formulae are the constraints
- $\Pi \subseteq \Sigma$: set of predicate symbols definable by a program
  - ◇ Atom: $p(t_1, t_2, \ldots, t_n)$, where $p \in \Pi$ and $t_1, t_2, \ldots, t_n$ are terms
  - ◇ *Primitive* constraint: $p(t_1, t_2, \ldots, t_n)$, where $t_1, t_2, \ldots, t_n$ are terms and $p \in \Sigma$ is a predicate symbol
  - ◇ Constraint: (first–order) formula built from primitive constraints
- The class of constraints will vary (generally only a subset of formulas are considered constraints)
- A **CLP program** is a collection of rules of the form $a \leftarrow b_1, \ldots, b_n$ where a is an atom and the b_i's are atoms or constraints
- A fact is a rule $a \leftarrow c$ where c is a constraint
- A goal (or query) G is a conjunction of constraints and atoms

A case study: CLP($\Re$)

- CLP($\Re$): language based on Prolog + constraint solving over the reals ($\mathcal{R}_{Lin}$)
  - ◇ Same execution strategy as standard Prolog (depth–first, left–to–right)
  - ◇ Allows linear equations and disequations over the reals
  - ◇ Linear constraints are solved;
    non-linear constraints are *passive*: delayed until linear or simple checks:
    - `X*Y = 7` becomes linear when `X` is assigned a definite value
    - `X*X+2*X+1 = 0` becomes a check when `X` is assigned a definite value
  - ◇ Prolog arithmetic is subsumed by constraint solving
- Note: CLP($\Re$) is really CLP(($\Re, \mathcal{FT}$)) — $\mathcal{FT}$ is often omitted.

- Supported in modern Prologs *coexisting* with the ISO primitives `is/2, >/2` etc.
- In Ciao, via the `clpr` package:
  - ◇ Uses `.=.`, `.>.`, etc. to distinguish the `clpr` constraints from the ISO-Prolog arithmetic primitives.
  - ◇ I.e., `X .=.Y + 5, Y .>.1` vs. `X is Y +5, Y >1`

Linear Equations (CLP($\Re$) package)

- Vector × vector multiplication (dot product):

  $\cdot : \Re^n \times \Re^n \longrightarrow \Re$

  $(x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) = x_1 \cdot y_1 + \cdots + x_n \cdot y_n$

- Vectors represented as lists of numbers

```
:- use_package(clpr).
prod([], [], Result) :- Result .=. 0.
prod([X|Xs], [Y|Ys], Result) :-
    Result .=. X * Y + Rest, prod(Xs, Ys, Rest).
```

- Unification becomes constraint solving!

```
?- prod([2, 3], [4, 5], K).
K .=. 23
?- prod([2, 3], [5, X2], 22).
X2 .=. 4
?- prod([2, 7, 3], [Vx, Vy, Vz], 0).
Vx .=. -1.5*Vz - 3.5*Vy
```

- Any computed answer is, in general, an equation over the variables in the query

Systems of Linear Equations (CLP($\Re$))

- Can we solve systems of equations? E.g.,

$$\begin{aligned} 3x + y &= 5 \\ x + 8y &= 3 \end{aligned}$$

- Write them down at the top level prompt:

```
?- prod([3, 1], [X, Y], 5), prod([1, 8], [X, Y], 3).
X .=. 1.6087, Y .=. 0.173913
```

- A more general predicate can be built mimicking the mathematical vector notation $A \cdot x = b$:

```
system(_Vars, [], []).
system(Vars, [Co|Coefs], [Ind|Indeps]) :-
    prod(Vars, Co, Ind),
    system(Vars, Coefs, Indeps).
```

- We can now express (and solve) equation systems

```
?- system([X, Y], [[3, 1],[1, 8]],[5, 3]).
X .=. 1.6087, Y .=. 0.173913
```

Non–linear Equations (CLP(ℜ))

- Non–linear equations are delayed

```
?- sin(X) .=. cos(X).
sin(X) .=. cos(X)
```

- This is also the case if there exists some procedure to solve them

```
?- X*X + 2*X + 1 .=. 0.
-2*X - 1 .=. X * X
```

- Reason: no general solving technique is known. CLP(ℜ) solves only linear (dis)equations.

- Once equations become linear, they are handled properly:

```
?- X .=. cos(sin(Y)), Y .=. 2+Y*3.
Y .=. -1, X .=. 0.666367
```

- Disequations are solved using a modified, incremental Simplex

```
?- X + Y .=<. 4, Y .>=. 4, X .>=. 0.
Y .=. 4, X .=. 0
```

Fibonaci Revisited (Prolog)

- Fibonaci numbers:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_{n+2} &= F_{n+1} + F_n \end{aligned}$$

- (The good old) Prolog version:

```
fib(0, 0).
fib(1, 1).
fib(N, F) :-
        N > 1,
        N1 is N - 1,
        N2 is N - 2,
        fib(N1, F1),
        fib(N2, F2),
        F is F1 + F2.
```

- Can only be used with the first argument instantiated to a number

Fibonaci Revisited (CLP($\Re$))

- CLP($\Re$) package version: syntactically similar to the previous one:

```
:- use_package(clpr).
fib(N,N) :- N .=. 0.
fib(N,N) :- N .=. 1.
fib(N,R) :- N .>. 1, F1 .>=. 0, F2 .>=. 0,
            N1 .=. N - 1, N2 .=. N - 2,
            fib(N1,F1), fib(N2,F2),
            R .=. F1 + F2.
```

- Note all constraints included in program (`F1 >=0, F2 >=0`) – good practice!
- Only real numbers and equations used (no data structures, no other constraint system): “pure CLP($\Re$)”
- Semantics greatly enhanced! E.g.:

```
?- fib(N, F).
F .=. 0, N .=. 0  ;
F .=. 1, N .=. 1  ;
F .=. 1, N .=. 2  ;
F .=. 2, N .=. 3  ;
```

Analog RLC circuits (CLP($\Re$))

- Analysis and *synthesis* of analog circuits
- RLC network in steady state
- Each circuit is composed either of:
  - ◇ A simple component, or
  - ◇ A connection of simpler circuits
- For simplicity, we will suppose subnetworks connected only in parallel and series $\longrightarrow$ Ohm's laws will suffice (other networks need global, i.e., Kirchoff's laws)
- We want to relate the current (`I`), voltage (`V`) and frequency (`W`) in steady state
- Entry point: `circuit(C, V, I, W)` states that:
  across the network `C`, the voltage is `V`, the current is `I` and the frequency is `W`
- `V` and `I` **must** be modeled as complex numbers (the imaginary part takes into account the angular frequency)
- Note that Herbrand terms are used to provide data structures

Analog RLC circuits (CLP($\Re$))

- Complex number $X + Yi$ modeled as `c(X, Y)`
- Basic operations:

```
:- use_package(clpr).

c_add(c(Re1,Im1), c(Re2,Im2), c(Re12,Im12)) :-
     Re12 .=. Re1+Re2,
     Im12 .=. Im1+Im2.

c_mult(c(Re1, Im1), c(Re2, Im2), c(Re3, Im3)) :-
       Re3 .=. Re1 * Re2 - Im1 * Im2,
       Im3 .=. Re1 * Im2 + Re2 * Im1.
```

(equality is `c_equal(c(R, I), c(R, I))`, can be left to [extended] unification)

Analog RLC circuits (CLP($\Re$))

- Circuits in series:

```
circuit(series(N1, N2), V, I, W) :-
        c_add(V1, V2, V),
        circuit(N1, V1, I, W),
        circuit(N2, V2, I, W).
```

- Circuits in parallel:

```
circuit(parallel(N1, N2), V, I, W) :-
        c_add(I1, I2, I),
        circuit(N1, V, I1, W),
        circuit(N2, V, I2, W).
```

Analog RLC circuits (CLP($\Re$))

Each basic component can be modeled as a separate unit:

- Resistor: $V = I * (R + 0i)$

```
circuit(resistor(R), V, I, _W) :-
    c_mult(I, c(R, 0), V).
```

- Inductor: $V = I * (0 + WL\ i)$

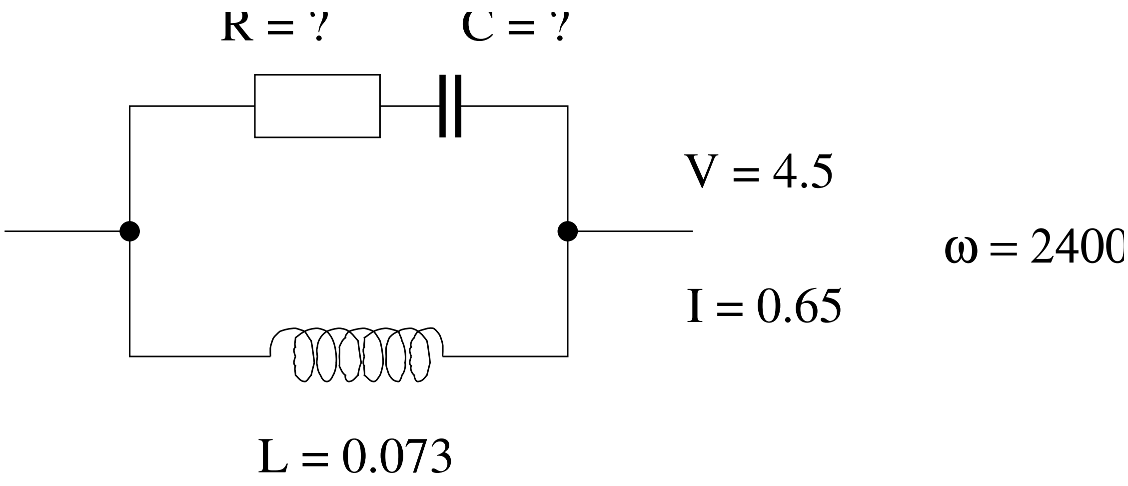
```
circuit(inductor(L), V, I, W) :-
    Im .=. W * L,
    c_mult(I, c(0, Im), V).
```

- Capacitor: $V = I * (0 - \frac{1}{WC}\ i)$

```
circuit(capacitor(C), V, I, W) :-
    Im .=. -1 / (W * C),
    c_mult(I, c(0, Im), V).
```

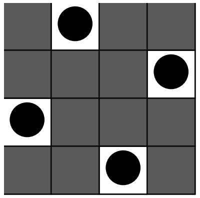
Analog RLC circuits (CLP(ℜ))

- Example:



```
?- circuit(parallel(inductor(0.073),
                    series(capacitor(C), resistor(R))),
           c(4.5, 0), c(0.65, 0), 2400).

R .=. 6.91229, C .=. 0.00152546

?- circuit(C, c(4.5, 0), c(0.65, 0), 2400).
```

The N Queens Problem

- Problem:
  place `N` chess queens in a `N` × `N` board such that they do not attack each other
- Data structure: a list holding the column position for each row
- The final solution is a permutation of the list `[1, 2, ..., N]`
- E.g.: the solution  is represented as `[2, 4, 1, 3]`
- General idea:
  - ◇ Start with partial solution
  - ◇ Nondeterministically select new queen
  - ◇ Check safety of new queen against those already placed
  - ◇ Add new queen to partial solution if compatible; start again with new partial solution

The N Queens Problem in Prolog

```
queens(N, Qs) :- queens_list(N, Ns),  % E.g., Ns=[4,3,2,1]
                 queens(Ns, [], Qs).
```

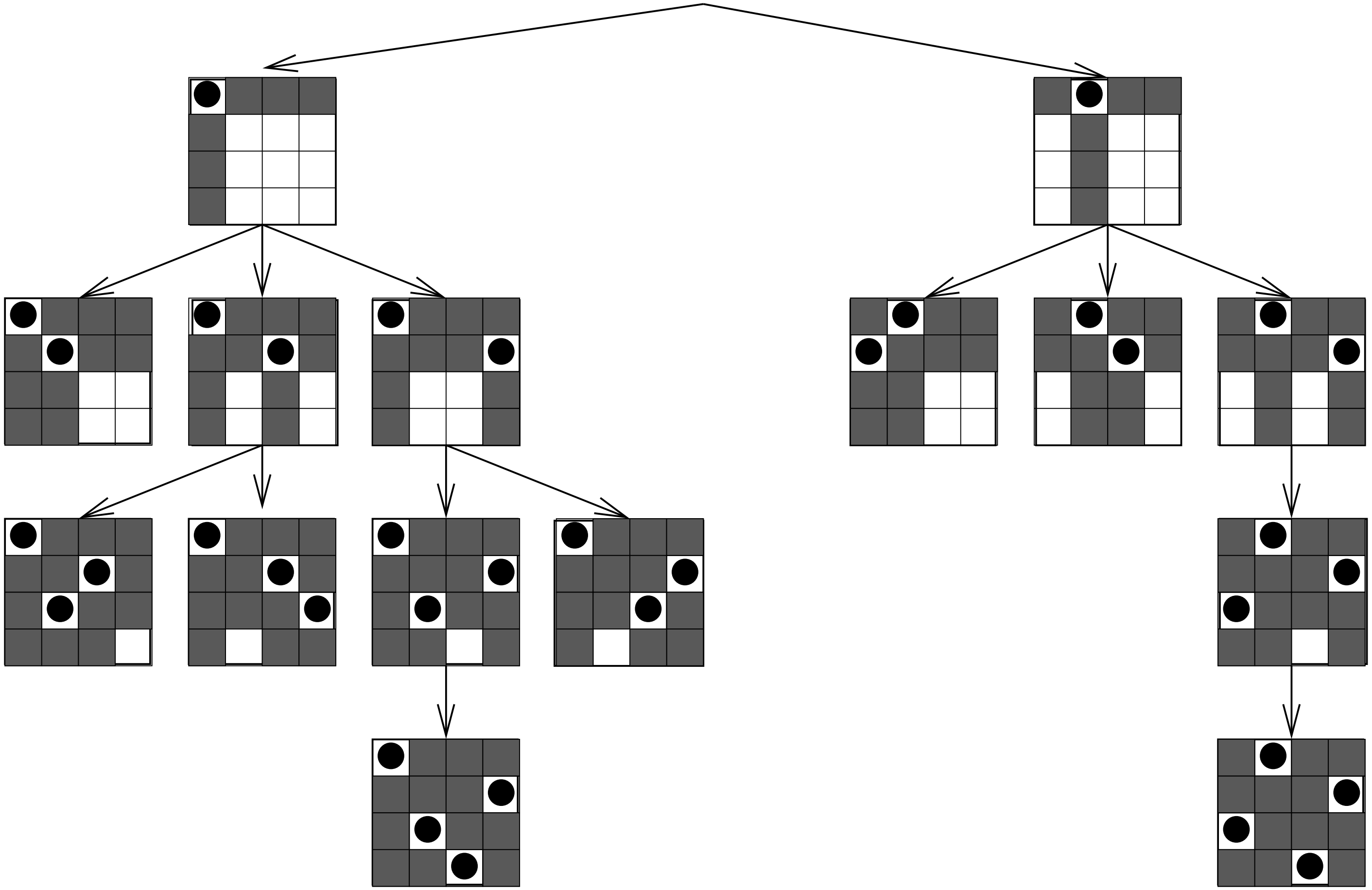
```
queens([], Qs, Qs).                   % All queens placed!
queens(Unplaced, Placed, Qs) :-
    select(Unplaced, Q, NewUnplaced), % E.g. Q=4, NewU=[3,2,1]
    no_attack(Placed, Q, 1),            % Fail if attack
    queens(NewUnplaced, [Q|Placed], Qs).% OK->Choose next q
```

```
no_attack([], _Queen, _Nb).
no_attack([Y|Ys], Queen, Nb) :-
    Queen =\=  Y + Nb,  Queen  =\= Y - Nb,
    Nb1 is Nb + 1, no_attack(Ys, Queen, Nb1).
```

```
select([X|Ys], X, Ys).
select([Y|Ys], X, [Y|Zs]) :- select(Ys, X, Zs).
```

```
queens_list(0, []).
queens_list(N, [N|Ns]) :-
    N > 0, N1 is N - 1, queens_list(N1, Ns).
```

The N Queens Problem in Prolog - search space



The N Queens Problem in CLP($\Re$) (in Ciao clpr syntax)

```
:- use_package(clpr).
queens(N,Qs) :- constrain_values(N,N,Qs), place_queens(N,Qs).
```

```
constrain_values(0, _N, []).       % Constrain before placing
constrain_values(I, N, [X|Xs]) :-
    I .>. 0,
    X .>. 0, X .<=. N, % All queens between 0 and N
    I1 .=. I - 1,
    constrain_values(I, N, Xs), no_attack(Xs, X, 1).
```

```
no_attack([], _Queen, _Nb).        % Identical to Prolog version
no_attack([Y|Ys], Queen, Nb) :-    % but using constraints
    Queen .<>. Y + Nb,  Queen .<>. Y - Nb,
    Nb1 .=. Nb + 1, no_attack(Ys, Queen, Nb1).
```

```
place_queens(0, _).
place_queens(N, Q) :-
    N .>. 0,
    member(N, Q),
    N1 .=. N - 1, place_queens(N1, Q).
```

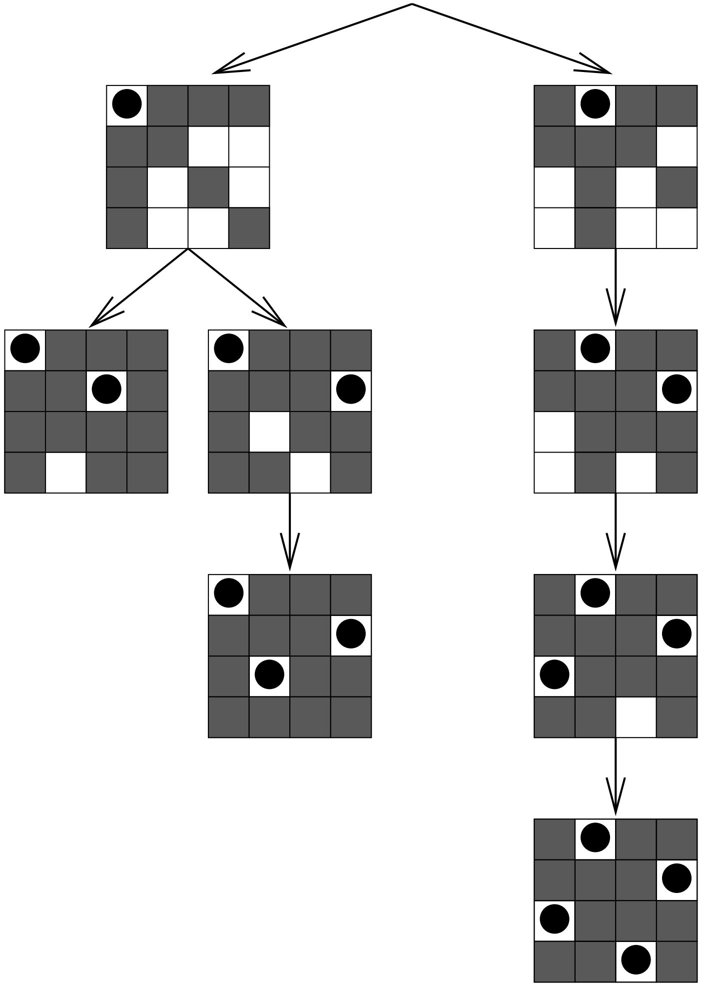
The N Queens Problem in CLP($\Re$)

- This last program can attack the problem in its most general instance:

```
?- queens(N,L).
L = [], N .=. 0 ;
L = [1], N .=. 1 ;
L = [2, 4, 1, 3], N .=. 4 ;
L = [3, 1, 4, 2], N .=. 4 ;
L = [5, 2, 4, 1, 3], N .=. 5 ;
L = [5, 3, 1, 4, 2], N .=. 5 ;
L = [3, 5, 2, 4, 1], N .=. 5 ;
L = [2, 5, 3, 1, 4], N .=. 5
...
```

- Remark: Herbrand terms used to build the data structures
- But also used as constraints (e.g., length of already built list `Xs` in `no_attack(Xs, X, 1)`)
- Note that in fact we are using both $\Re$ and $\mathcal{FT}$

The N Queens Problem in CLP($\Re$) – search space



The N Queens Problem in CLP($\Re$)

- CLP($\Re$) generates internally a set of equations for each board size

```
?- constrain_values(4, 4, Qs).
Qs = [_A,_B,_C,_D],
nonzero(_E),   _A.=<.4.0,    _E.=.3.0+_A-_D,
nonzero(_F),   _A.>.0,       _F.=. -3.0+_A-_D,
nonzero(_G),   _B.=<.4.0,    _G.=.2.0+_A-_C,
nonzero(_H),   _B.>.0,       _H.=. -2.0+_A-_C,
nonzero(_I),   _C.=<.4.0,    _I.=.1+_A-_B,
nonzero(_J),   _C.>.0,       _J.=. -1+_A-_B,
nonzero(_K),   _D.=<.4.0,    _K.=.2.0+_B-_D,
nonzero(_L),   _D.>.0,       _L.=. -2.0+_B-_D,
nonzero(_M),                 _M.=.1+_B-_C,
nonzero(_N),                 _N.=. -1+_B-_C,
nonzero(_O),                 _O.=.1+_C-_D,
nonzero(_P),                 _P.=. -1+_C-_D      ?
```

- `place_queens(4,[_A,_B,_C,_D])` adds all possible queens in `[_A,_B,_C,_D]`.

The N Queens Problem in CLP($\Re$)

- Constraints are (incrementally) simplified as new queens are added

```
?- constrain_values(4, 4, Qs), Qs = [3,1|_].
Qs = [_A,_B,_C,_D],
nonzero(_E),     _A.=.3.0,        _E.=.6.0-_D,
nonzero(_F),     _B.=.1.0,        _F.=. -_D,
nonzero(_G),     _C.=<.4.0,       _G.=.5.0-_C,
nonzero(_H),     _C.>.0,          _H.=.1.0-_C,
nonzero(_I),     _D.=<.4.0,       _I.=.3.0-_D,
nonzero(_J),     _D.>.0,          _J.=. -1.0-_D,
nonzero(_K),                      _K.=.2.0-_C,
nonzero(_L),                      _L.=. -_C,
nonzero(_M),                      _M.=.1+_C-_D,
nonzero(_N),                      _N.=. -1+_C-_D ?
```

- Bad choices are rejected using constraint consistency:

```
?- constrain_values(4, 4, Qs), Qs = [3,2|_].
no
```

Finite Domains (I)

- A *finite domain* constraint solver associates each variable with a finite subset of $\mathcal{Z}$
- Example: $E \in \{-123, -10..4, 10\}$
  Can be represented as, e.g., `E :: [-123, -10..4, 10]` [Eclipse notation]
  or as `E in -123\/(-10..4)\/10` [Ciao notation]
- We can:
  - ◇ Establish the *domain* of a variable (`in`).
  - ◇ Perform arithmetic operations (`+`, `-`, `*`, `/`) on the variables
  - ◇ Establish linear relationships among arithmetic expressions (`#=`, `#<`, `#=<`)
- These operations / relationships narrow the domains of the variables
- **Note:** In Ciao this functionality is loaded with a
  `:- use_package(clpfd).`
  directive in the source code –or, equivalently, adding in the module declaration:
  `:- module(_, ..., [clpfd]).`

Finite Domains (II)

Examples:

```
?- X #= A + B, A in 1..3, B in 3..7.
X in 4..10, A in  1..3, B in  3..7
```

- The respective minimums and maximums are added
- There is no unique solution

```
?- X #= A - B, A in 1..3, B in 3..7.
X in -6..0, A in 1..3, B in 3..7
```

- The min value of `X` is the min value of `A` minus the max value of `B`
- (Similar for the maximum values)

```
?- X #= A - B, A in  1..3, B in  3..7, X #>= 0.
    A = 3, B = 3, X = 0
```

- Putting more constraints results in a unique solution.

Finite Domains (III)

Some useful primitives in finite domains:

- `domain(Variables, Min, Max)`: A shorthand for several `in` constraints
- `labeling(Options, VarList)`:
  - ◇ instantiates variables in `VarList` to values in their domains
  - ◇ `Options` dictates the search order

```
?- domain([X, Y, Z],1,1000), X*X+Y*Y #= Z*Z, X #>= Y,
   labeling([],[X,Y,Z]).
X = 4, Y = 3, Z = 5,
X = 8, Y = 6, Z = 10,
X = 12, Y = 5, Z = 13,
...
```

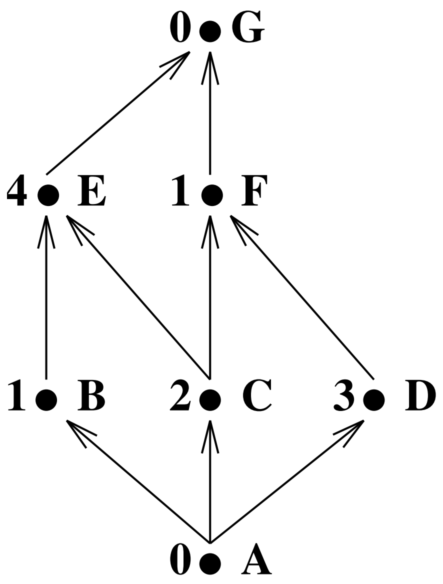
- `minimize(G, X)`: solve `G` minimizing the value of variable `X`
- This can be used to minimize (c.f., maximize) a solution

A classic example: SEND MORE MONEY

```
%     S E N D
%   + M O R E
%   ---------
%   M O N E Y

:- use_package(clpfd).

smm([S,E,N,D,M,O,R,Y]) :-
    domain([S,E,N,D,M,O,R,Y], 0, 9),    % All digits 0..9
    0 #< S, 0 #< M,                     % No leftmost zeros
    all_different([S,E,N,D,M,O,R,Y]),   % All digits different
              S*1000 + E*100 + N*10 + D +    %
              M*1000 + O*100 + R*10 + E #=   % Arith. constr.
    M*10000 + O*1000 + N*100 + E*10 + Y,     %
    labeling([], [S,E,N,D,M,O,R,Y]).    % Instantiate variables
```

A Project Management Problem (I)

- The job whose dependencies and task lengths are given by this graph...



... should be finished in 10 time units or less.

- Constraints:

```
pn1(A,B,C,D,E,F,G) :-
    domain([A,B,C,D,E,F,G], 0, 10),
    A #>= 0, G #=< 10,
    B #>= A, C #>= A, D #>= A,
    E #>= B + 1, E #>= C + 2,
    F #>= C + 2, F #>= D + 3,
    G #>= E + 4, G #>= F + 1.
```

A Project Management Problem (II)

- Query:

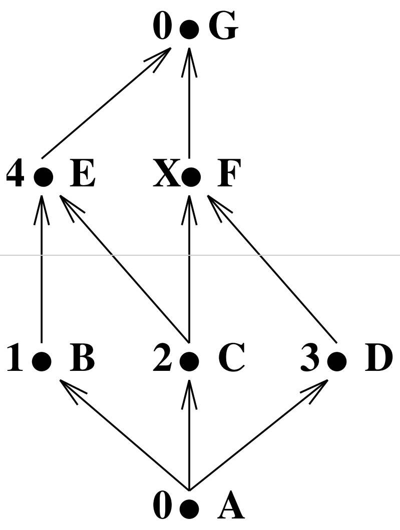
```
?- pn1(A,B,C,D,E,F,G).
A in 0..4, B in 0..5, C in 0..4,
D in 0..6, E in 2..6, F in 3..9, G in 6..10.
```

- Note the slack of the variables
- Some additional constraints must be respected as well, but are not shown by default
- Minimize the total project time:

```
?- minimize(pn1(A,B,C,D,E,F,G), G).
    A = 0, B in 0..1, C = 0, D in 0..2,
    E = 2, F in 3..5, G = 6
```

- Variables without slack represent critical tasks

A Project Management Problem (III)



- An alternative setting:

- We can accelerate task **F** at some cost

```
pn2(A, B, C, D, E, F, G, X) :-
    domain([A,B,C,D,E,F,G,X], 0, 10),
    A #>= 0, G #=< 10,
    B #>= A, C #>= A, D #>= A,
    E #>= B + 1, E #>= C + 2,
    F #>= C + 2, F #>= D + 3,
    G #>= E + 4, G #>= F + X.
```

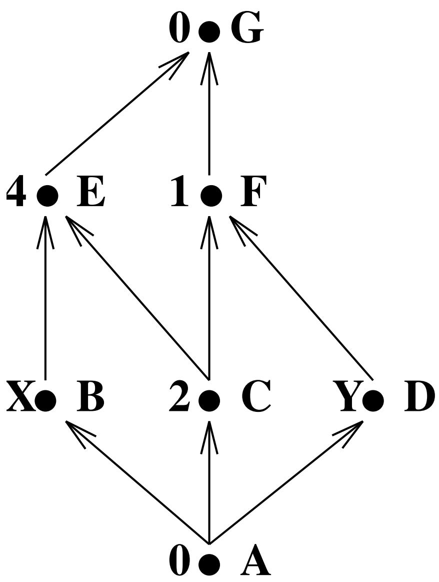
- We do not want to accelerate it more than needed!

→ minimize `G` and maximize `X`.

```
    A = 0, B in 0..1, C = 0, D = 0,
    E = 2, F = 3, G = 6, X = 3.
```

A Project Management Problem (IV)

- We have two independent tasks **B** and **D** whose lengths are not fixed:



- We can finish any of **B**, **D** in 2 time units at best
- Some shared resource disallows finishing *both* tasks in 2 time units: they will take 6 time units

A Project Management Problem (V)

- Constraints describing the net:

```
pn3(A,B,C,D,E,F,G,X,Y) :-
    domain([A,B,C,D,E,F,G,X,Y], 0, 10),
    A #>= 0, G #=< 10,
    X #>= 2, Y #>= 2, X + Y #= 6,
    B #>= A, C #>= A, D #>= A,
    E #>= B + X, E #>= C + 2,
    F #>= C + 2, F #>= D + Y,
    G #>= E + 4, G #>= F + 1.
```

- Query:

```
?- minimize(pn3(A,B,C,D,E,F,G,X,Y),G).
A = 0, B = 0, C = 0, D in 0..1, E = 2,
F in 4..5, X = 2, Y = 4, G = 6
```

- I.e., we must devote more resources to task `B`
- All tasks but **F** and **D** are critical now
- Sometimes, `minimize/2` not enough to provide best solution (pending constr.):

```
?- minimize(pn3(A,B,C,D,E,F,G,X,Y),G), labeling([],[D,F]).
```

The N-Queens Problem Using Finite Domains (in Ciao clpfd syntax)

- By far, the fastest implementation

```
:- use_package(clpfd).
queens(N, Qs, Type) :-              % Type is labeling strategy
    constrain_values(N, N, Qs),     % Constrain before placing
    all_different(Qs),              % Using built-in constraint
    labeling(Type,Qs).              % Labeling places the queens
```

```
constrain_values(0, _N, []).
constrain_values(N, NMax, [X|Xs]) :-
    N > 0, N1 is N - 1, X in 1 .. NMax, % Limits X values
    constrain_values(N1, NMax, Xs), no_attack(Xs, X, 1).
```

```
no_attack([], _Queen, _Nb).         % Same as CLP(R) version
no_attack([Y|Ys], Queen, Nb) :-     % but using clpfd primitives
    Queen #= Y + Nb, Queen #= Y - Nb, Nb1 is Nb + 1,
    no_attack(Ys, Queen, Nb1).
```

- Query: `?- queens(20, Q, [ff]).` (`Type` is the type of labeling desired.)

```
Q = [1,3,5,14,17,4,16,7,12,18,15,19,6,10,20,11,8,2,13,9] ?
```

CLP($\mathcal{FT}$) (a.k.a. Logic Programming)

- Equations over Finite Trees
- Check that two trees are isomorphic (same elements in each level)

```
iso(Tree, Tree).
iso(t(R, I1, D1), t(R, I2, D2)) :-
    iso(I1, D2),
    iso(D1, I2).

?- iso(t(a, b, t(X, Y, Z)), t(a, t(u, v, W), L)).
L=b, X=u, Y=v, Z=W ? ;
L=b, X=u, Y=W, Z=v ? ;
L=b, W=t(_C,_B,_A), X=u, Y=t(_C,_A,_B), Z=v ? ;
L=b, W=t(_E,t(_D,_C,_B),_A), X=u, Y=t(_E,_A,t(_D,_B,_C)),
  Z=v ?
```

CLP($\mathcal{WE}$)

- Equations over finite strings
- Primitive constraints: concatenation (.), string length (::)
- Find strings meeting some property:

```
?- "123".Z = Z."231", Z::0.      ?- "123".Z = Z."231", Z::3.
no                               no

?- "123".Z = Z."231", Z::1.      ?- "123".Z = Z."231", Z::4.
Z = "1"                          Z = "1231"

?- "123".Z = Z."231", Z::2.
no
```

- These constraint solvers are very complex
- Often incomplete algorithms are used

CLP(($\mathcal{WE}$, $\mathcal{Q}$))

- Word equations plus arithmetic over $\mathcal{Q}$ (rational numbers)
- Prove that the sequence $x_{i+2} = |x_{i+1}| - x_i$ has a period of length 9 (for any starting x_0, x_1)
- Strategy: describe the sequence, try to find a subsequence such that the period condition is violated
- Sequence description (syntax is Prolog III slightly modified):

```
seq(<Y, X>).                          abs(Y, Y) :- Y >= 0.
seq(<Y1 - X, Y, X>.U) :-              abs(Y,-Y) :- Y < 0.
    seq(<Y, X>.U)
    abs(Y, Y1).
```

- Query: *Is there any 11–element sequence such that the 2–tuple initial seed is different from the 2–tuple final subsequence (the seed of the rest of the sequence)?*

```
?- seq(U.V.W), U::2, V::7, W::2, U#W.
fail
```

Summarizing

- **In general:**
  - ◇ Data structures (Herbrand terms) for free
  - ◇ Each logical variable may have constraints associated with it (and with other variables)
- **Problem modeling :**
  - ◇ Rules represent the problem at a high level
    - * Program structure, modularity
    - * Recursion used to set up constraints
  - ◇ Constraints encode problem conditions
  - ◇ Solutions also expressed as constraints
- **Combinatorial search problems:**
  - ◇ CLP languages provide backtracking: enumeration is easy
  - ◇ Constraints keep the search space manageable
- **Tackling a problem:**
  - ◇ Keep an open mind: often new approaches possible

Complex Constraints

- Some complex constraints allow expressing simpler constraints
- May be operationally treated as passive constraints
- E.g.: cardinality operator $\#(L, [c_1, \ldots, c_n], U)$ meaning that the number of true constraints lies between L and U (which can be variables themselves)
  - ◇ If $L = U = n$, all constraints must hold
  - ◇ If $L = U = 1$, one and only one constraint must be true
  - ◇ Constraining $U = 0$, we force the conjunction of the negations to be true
  - ◇ Constraining $L > 0$, the disjunction of the constraints is specified
- Disjunctive constructive constraint: $c_1 \vee c_2$
  - ◇ If properly handled, avoids search and backtracking
  - ◇ E.g.:

$$\begin{aligned} nz(X) &\leftarrow X > 0. \\ nz(X) &\leftarrow X < 0. \end{aligned}$$

$$nz(X) \leftarrow X < 0 \vee X > 0.$$

Other Primitives

- CLP($\mathcal{X}$) systems usually provide additional primitives
- E.g.:
  - `enum(X)` enumerates `X` inside its current domain
  - `maximize(X)` (c.f. `minimize(X)`) works out maximum (minimum value) for `X` under the active constraints
  - `delay Goal until Condition` specifies when the variables are instantiated enough so that `Goal` can be effectively executed
    - Its use needs deep knowledge of the constraint system
    - Also widely available in Prolog systems
    - Not really a constraint: control primitive

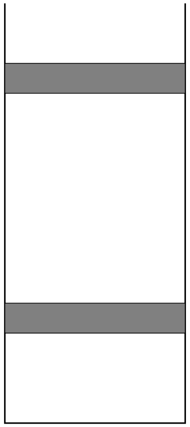
Implementation Issues: Satisfiability

- Algorithms must be *incremental* in order to be practical
- Incrementality refers to the performance of the algorithm
- It is important that algorithms to decide satisfiability have a good average case behavior
- Common technique: use a *solved form* representation for satisfiable constraints
- Not possible in every domain
- E.g. in $\mathcal{FT}$ constraints are represented in the form $x_1 = t_1(\tilde{y}), \ldots, x_n = t_n(\tilde{y})$, where
    - ◇ each $t_i(\tilde{y})$ denotes a term structure containing variables from $\tilde{y}$
    - ◇ no variable x_i appears in $\tilde{y}$

(i.e., idempotent substitutions, guaranteed by the unification algorithm)

Implementation Issues: Backtracking in CLP($\mathcal{X}$)

- Implementation of backtracking more complex than in Prolog
- Need to record changes to constraints
- Constraints typically stored as an association of variable to expression
- Trailing expressions is, in general, costly: cannot be stored at every change
- Avoid trailing when there is no choice point between two successive changes
- A standard technique: use *time stamps* to compare the age of the choice point with the age of the variable at the time of last trailing



X<9, Y=5, Z=4, W=1 **trail W, timestamp it**

X<Y+4, Y=4+W, Z=4 **trail X, Y, Z, timestamp them**

X<Y+Z, Y=Z+W **timestamp X, Y, Z, W**

Implementation Issues: Extensibility

- Constraint domains often implemented now in Prolog-based systems using:
  - ◇ Attributed variables [Neumerkel,Holzbaur]:
    - * Provide a hook into unification.
    - * Allow attaching an *attribute* to a variable.
    - * When unification with that variable occurs, user-defined code is called.
    - * Used to implement constraint solvers (and other applications, e.g., distributed execution).
  - ◇ Constraint handling rules (CHRs):
    - * Higher-level abstraction.
    - * Allows defining propagation algorithms (e.g., constraint solvers) in a high-level way.
    - * Often translated to attributed variable-based low-level code.

Attributed Variables Example: Freeze

- Primitives:
  - ◇ `attach_attribute(X,C)`
  - ◇ `get_attribute(X,C)`
  - ◇ `detach_attribute(X)`
  - ◇ `update_attribute(X,C)`
  - ◇ `verify_attribute(C,T)`
  - ◇ `combine_attributes(C1,C2)`
- *Example:* Freeze

```
freeze( X, Goal) :-
  attach_attribute( V, frozen(V,Goal)),
  X = V.

verify_attribute( frozen(Var,Goal), Value) :-
  detach_attribute( Var),
  Var = Value,
  call(Goal).

combine_attributes( frozen(V1,G1), frozen(V2,G2)) :-
  detach_attribute( V1),
  detach_attribute( V2),
  V1 = V2,
  attach_attribute( V1, frozen(V1,(G1,G2))).
```

Programming Tips

- Over-constraining:
  - Seems to be against general advice “do not perform extra work”, but can actually cut more search space
  - Specially useful if *infer* is weak
  - Or else, if constraints outside the domain are being used
- Use control primitives (e.g., cut) very sparingly and carefully
- Determinacy is more subtle, (partially due to constraints in non–solved form)
- Choosing a clause does not preclude trying other exclusive clauses (as with Prolog and plain unification)
- Compare:

```
max(X,Y,X) :- X .>. Y.                    ?- max(X, Y, Z).
max(X,Y,Y) :- X .<=. Y.                   Z .=. X, Y .<. X ;
```

with

```
max(X,Y,X) :- X .>. Y, !.                 ?- max(X, Y, Z).
max(X,Y,Y) :- X .<=. Y.                   Z .=. X, Y .<. X
```

CLP Systems

- As mentioned before, CLP defines a class of languages obtained by
  - Specifying the particular constraint system(s)
  - Specifying the *Computation* and *Selection* rules
- Most practical systems include also the Herbrand domain with “=”, but then add different domains and/or solver algorithms
- Most use the *Computation* and *Selection* rules of Prolog

Some Classic CLP Systems

- **CLP($\Re$):**
  - ◇ Linear arithmetic over reals ($=, \leq, >$) – CLP(R)
    Incremental Gaussian elimination and incremental Simplex
- **PrologIII:**
  - ◇ CLP(R)
  - ◇ Boolean ($=$), 2-valued Boolean Algebra – CLP(B)
  - ◇ Infinite (rational) trees ($=, \neq$)
  - ◇ Equations over finite strings – CLP(WE)
- **CHIP** (and its successor: the **ILOG** library):
  - ◇ CLP(FD), CLP(B), CLP(Q)
  - ◇ User–defined constraints and solver algorithms
- **BNR-Prolog / CLP(BNR):**
  - ◇ Arithmetic over reals (closed intervals); CLP(FD), CLP(B).
- **RISC–CLP:**
  - ◇ Arithmetic constraints over reals, also non-linear
    (using Presburger arithmetic)

Some Current CLP Systems

- **clp(`FD`)/gprolog:**
  - ◇ CLP(FD).
- **SICStus:**
  - ◇ CLP(R), CLP(Q), CLP(FD)
  - ◇ Attributed variables and CHR for adding domains.
- **ECL^{i}PS^e:**
  - ◇ CLP(R), CLP(Q), CLP(FD).
- **SWI:**
  - ◇ CLP(R), CLP(Q), CLP(FD), CLP(B).
  - ◇ Attributed variables and CHR for additional domains.
- **Ciao:**
  - ◇ CLP(R), CLP(Q), CLP(FD).
  - ◇ Attributed variables and CHR for additional domains.
  - ◇ Different domains can be activated on a per-module basis (packages).

→ Most Prolog systems now support constraints!

Some origins and other instances

- Ancestors:
  - ◇ SKETCHPAD (1963), Waltz's algorithm (1965?), THINGLAB (1981), MACSYMA (1983), ...
- Constraints in logic languages: – the origin of "constraint programming":
  - ◇ General theory developed (Jaffar and Lassez '97).
  - ◇ First, standalone systems developed: `clpr`, `CHIP`, ...
  - ◇ Later, included in mainstream Prolog implementations.
  - ◇ Has given rise to a whole research area!
- Constraints in imperative languages:
  - ◇ Equation solving libraries (ILOG, GECODE, ...)
  - ◇ Timestamping of variables: `x := x + 1` $\leftrightarrow x_{i+1} := x_i + 1$ (similar to iterative methods in numerical analysis)
- Constraints in functional languages, via extensions:
  - ◇ Evaluation of expressions including free variables.
  - ◇ *Absolute Set Abstraction.*